DISSERTATION

Spectral Theory of Modular Operators for von Neumann Algebras and Related Inverse Problems

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Spectral Theory of Modular Operators for von Neumann Algebras and Related Inverse Problems

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Referat

In dieser Arbeit werden die Modularobjekte zu zyklischen und separierenden Vektoren für von-Neumann-Algebren untersucht. Besondere Beachtung erfahren dabei die Modularoperatoren und deren Spektraleigenschaften. Diese Eigenschaften werden genutzt, um Klassifikationen für Lösungen einiger inverser Probleme der Modulartheorie anzugeben. Im ersten Teil der Arbeit wird zunächst der Zusammenhang zwischen dem zyklischen und separierenden Vektor und seinen Modularobjekten mit Hilfe (verallgemeinerter) Spurvektoren für halbendliche und Typ III_λ Algebren $(0 < \lambda < 1)$ näher untersucht. Diese Untersuchungen erlauben es, das Spektrum der Modularoperatoren für Typ I Algebren anzugeben. Dazu werden die Begriffe *zentraler Eigenwert* und *zentrale Vielfachheit* eingeführt. Weiterhin ergibt sich, dass die Modularoperatoren durch ihre Spektraleigenschaften eindeutig charakterisiert sind. Modularoperatoren für Typ I_n Algebren sind genau die *n-zerlegbaren* Operatoren, die *multiplikatives, zentrales Spektrum vom Typ I_n* besitzen. Ähnliche Ergebnisse werden auch für Typ II und III_λ Algebren gewonnen unter der Vorausetzung, dass die zugehörigen Vektoren diagonalisierbar sind. Im zweiten Teil der Arbeit werden diese Ergebnisse exemplarisch auf ein inverses Problem der Modulartheorie angewendet. Dabei stellt sich heraus, dass die Begriffe zentraler Eigenwert und zentrale Vielfachheit Invarianten des inversen Problems sind und eine vollständige Klassifizierung seiner Lösungen unter obigen Voraussetzungen erlauben. Außerdem wird eine Klasse von Modularoperatoren untersucht, für die das inversese Problem nur ein oder zwei Lösungsklassen besitzt.

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Abstract

In this work modular objects of cyclic and separating vectors for von Neumann algebras are considered. In particular, the modular operators and their spectral properties are investigated. These properties are used to classify the solutions of some inverse problems in modular theory. In the first part of the work the correspondence between cyclic and separating vectors and their modular objects are considered for semifinite and type III_λ algebras ($0 < \lambda < 1$) in more detail, where (generalized) trace vectors are used. These considerations allow to compute the spectrum of modular operators for type I algebras. To this end, the notions of *central eigenvalue* and *central multiplicity* are introduced. Furthermore, it is stated that modular operators are uniquely determined by their spectral properties. Modular operators for type I_n algebras are exactly the n*-decomposable* operators, which possess *multiplicative central spectrum of type* I_n. Similar results are derived for type II and III_λ algebras under the assumption that the corresponding vectors are diagonalizable. In the second part of this work these results are applied to an inverse problem of modular theory. It comes out, that the central eigenvalues and central multiplicities are invariants of this inverse problem and that they give a complete classification of its solutions. Moreover, a class of modular operators is investigated, whose inverse problem possesses only one or two classes of solutions.

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Chapter 1

Introduction

1.1 Historical Remarks

Since its discovery modular theory has become an important and powerful tool in the theory of operator algebras and quantum field theory. Although Tomita already distributed a preprint on his discovery at the Baton Rouge conference in 1967 [Tom67] modular theory became only known to a broader audience by the treatment of Takesaki [Tak70]. Therefore it is often called Tomita-Takesaki-Theory.

Already in the beginning of its development the similarity between some aspects of Tomita's theory and the approach of Haag, Hugenholtz, and Winnink to the thermodynamic equilibrium states [HHW67] was realized by Winnink and Takesaki (see the remark in [Con94, I.3]). This observation demonstrates the strong relation between modular theory and mathematical physics. Nevertheless, research in modular theory was mainly a subject of mathematical research in the first decade of its existence. Modular theory was then developed into an essential tool for the treatment of numerous problems in operator algebras. It contributed to a better understanding of the structure of factors, produced examples and refined their classification. It allows to introduce invariants for arbitrary factors which previously were only defined in some special cases. An outstanding example is Connes' classification of type III factors [Con73] which finally led to a complete classification of hyperfinite factors [Con76, Con85, Haa87]. An important tool in these investigations was the spectrum of modular operators.

Other names connected to the development of modular theory are Arveson, Haagerup, Landstad, Pedersen, Takesaki, Tomiyama, and others (see the preface of the book of Strătilă, [Str81]). Some mathematical applications of modular theory will be presented in §2.3 of this thesis.

Recently, modular theory has also become a tool in the index theory of subfactors, initiated by Jones [Jon83] and generalized to arbitrary factors by Kosaki [Kos86]. For further applications of modular theory and more details we refer to §2.3 and [Str81]. Moreover, Connes' book [Con94] is worth reading to those who are interested in the history of modular theory, especially in Connes' contributions to it.

Because of its great success in the structure theory of von Neumann algebras there were some attempts to generalize modular theory to other structures, e. g. to unbounded algebras [Ino98], to Poisson manifolds [Wei97], or to bimodules [Yam94].

Nowadays, research in modular theory is motivated mainly by its applications to quantum field theory. For in the algebraic approach to quantum field theory the so-called Reeh-Schlieder-Theorem [RS61] often ensures the applicability of modular theory. In some cases the modular objects are explicitly known and have a concrete physical meaning (cf. e. g. the Bisognano-Wichmann Theorem [BW75, BW76]). This allows to prove some important results in the theory of local observables, e. g. the PCT theorem. Furthermore, there is some hope that modular theory can help to distinguish theories with different dynamics and that it can serve as a selection criteria for physically relevant theories. Except for some sporadic earlier results, e. g. the Bisognano-Wichmann Theorem [BW75, BW76], the main development of modular theory as a tool in quantum field theory started in the early nineties. Contributors are Borchers, Brunetti, Buchholz, Guido, Longo, Summers, Wiesbrock, Wollenberg, and others (see [BW92] and the survey article by Borchers [Bor00]). Some applications will be presented in §2.4.

1.2 Main Results

Although there has been a continuing study of modular theory for 30 years many problems still remain unsolved. Most authors only consider the modular automorphism groups, which are the natural objects from the algebraic point of view. But if we are interested in the spatial structure of the theory, which is, for instance, sometimes required by physical applications we must investigate the modular operator and the modular conjugation (modular objects for short) themselves. The first aim of this thesis is hence to contribute to a better understanding of the modular objects, especially the modular operators. The main tool used for the investigation is the spectral theory of unbounded selfadjoint operators. Although many spectral properties of modular operators were examined in the structure theory of von Neumann algebras there seems to be no systematic study of spectral theory for general modular operators, except for some sporadic results. This dissertation is intended to close this gap at least partially. The results obtained by spectral theory will then be used to provide first insights into inverse problems related to modular theory which are inspired by physical applications.

The main results are contained in Chapter 4 and Chapter 6. In Chapter 4 we will develop the spectral theory of modular operators. To this end, we will introduce the notions of operators *n-decomposable with respect to an abelian algebra $\mathcal{B}$* and of operators which have *multiplicative central spectrum of type I_n* ($n \in \mathbb{N} \cup \{\infty\}$). To illustrate this notion, consider the case $\mathcal{B} = \mathbb{C}$. Roughly speaking, an operator which is n-decomposable with respect to $\mathbb{C}$ is an operator acting on a n^2-dimensional Hilbert space which possesses pure point spectrum, which is obviously a restriction only in the infinite-dimensional case ($n = \infty$).

An n-decomposable operator has multiplicative central spectrum of type I_n if the family of its eigenvalues can be written as the products of two families of numbers with n elements. For a precise definition we refer to Definition 4.2.10. Using this notion we will prove that a positive invertible operator is the modular operator corresponding to a cyclic and separating vector for a von Neumann algebra of type I_n with center $\mathcal{B}$ if and only if it is n-decomposable with respect to $\mathcal{B}$ and has multiplicative central spectrum of type I_n (Theorem 4.2.12).

Thus, all modular operators for type I algebras are characterized by their spectral properties. Similar characterizations are possible for modular operators corresponding to vectors which are *diagonalizable with respect to the center* in the type II and the type III_λ case $(0 \leq \lambda < 1)$ (Theorem 4.3.12 and Theorem 4.4.7). Diagonalizable vectors for type II algebras and type III_λ factors are, loosely speaking, cyclic and separating vectors having properties similar to the properties of cyclic and separating vectors for type I algebras (for details we refer to Definition 4.3.1).

These results are interesting in their own right and, at the same time, the basis of the investigation of inverse problems in modular theory. One of these inverse problems is concerned with the following task. Let u_0 be a fixed cyclic and separating vector for a fixed von Neumann algebra $\mathcal{M}_0$ with corresponding modular objects Δ_0 and J_0. Then the von Neumann algebra $\mathcal{M}_0$ is far from being the only von Neumann algebra with modular objects Δ_0 and J_0 corresponding to the same vector u_0. We would like to classify all von Neumann algebras $\mathcal{M}$ isomorphic to $\mathcal{M}_0$ which have the same modular objects Δ_0 and J_0 for the cyclic and separating vector u_0. A motivation for this problem and a precise formulation can be found in §5.1. The other problems are of a similar nature.

Due to a result of Wollenberg [Wol97] this inverse problem can be transformed into the study of the modular operators for the fixed algebra $\mathcal{M}_0$ which are unitarily equivalent to the given modular operator Δ_0. Hence, the results obtained in Chapter 4 can be applied.

In the type I case it is possible to completely classify the solutions of the inverse problem in terms of the central eigenvalues and central multiplicities of operators which are assigned to the solutions (Theorem 6.2.8). In the factor case the central eigenvalues and central multiplicities are the usual eigenvalues and multiplicities of selfadjoint operators. For the general definition see Definition 6.2.1.

The central eigenvalues and central multiplicities also allow a classification in the type II case. However, since we will use the results of Chapter 4 this is only possible for solutions which correspond to diagonalizable vectors (Theorem 6.3.8).

Furthermore, we will present a class of modular operators for semifinite factors, the so-called modular operators with *generic spectrum.* A modular operator possesses generic spectrum if its eigenvalues are, roughly speaking, randomly distributed (the precise definition is Definition 6.5.10). The inverse problems of these modular operators have at most two (simple) classes of solutions (Theorem 6.5.12).

1.3 Organisation

As for prerequisites, the reader is expected to be familiar with the general theory of von Neumann algebras which can be found in all standard books on operator algebras, e. g. [KR83] or [SZ79].

In Chapter 2 we recall the most important results of modular theory used in this thesis. This is done for the reader's convenience as well as for the ease of reference and to fix notations. Additionally, we give some insights into mathematical and physical applications of modular theory.

The examination of modular objects starts in Chapter 3 with the introduction of "generalized vectors" (a concept first introduced by Inoue and Karwowski [IK94]). This concept will prove useful in the remaining sections where we will obtain the general form of modular operators for semifinite algebras and for type III_λ factors ($0 \leq \lambda < 1$) (§3.2, §3.3, and §3.4.1) as well as for some special cases of type III_1 factors (§3.4.2). Most of these results seem to be more or less commonly known, especially those for semifinite factors. However, they can be found only partially in the literature. Moreover, the unified approach used here and some minor results seem to be new.

Chapter 4 contains the results on spectral theory of modular operators sketched in the previous section. The starting point are some remarks on the spectra of products of two commuting operators, where the first is affiliated with a von Neumann algebra and the second is affiliated with its commutant (§4.1). This is motivated by the observation that the modular operators are composed of such products. In §4.2-§4.4 the spectral theory of modular operators will be developed separately for the different types.

Chapter 5 contains an introduction to some inverse problems in modular theory. We will motivate and formulate them in §5.1, and we will prove some of their general properties in §5.2. Furthermore, we will define an equivalence relation in terms of which the solutions of the inverse problems will be classified, and we will present two simple classes of solutions (§5.3).

The results on the classification of the inverse problem are contained in Chapter 6. §6.1 is devoted to some general results for the semifinite case. The classification will be carried out in §6.2 for the type I algebras and in §6.3 for the type II algebras. Some remarks on the classification in the type III_λ case ($0 < \lambda < 1$) will be added in §6.4. We will close with the theorem on modular operators with generic spectrum (§6.5).

In the last chapter we will summarize the results, give an outlook on open problems and formulate suggestions for further developments of the theory.

In the Appendix technical results necessary for some of the proofs are collected.

1.4 Conventions and Notations

Hilbert spaces are always assumed to be separable. This means that the von Neumann algebras acting on them are separable (they possess a separable predual, they are countably decomposable). $L(\mathcal{H})$ denotes the set of all bounded linear operators on a Hilbert space $\mathcal{H}$. Scalar products $\langle\cdot|\cdot\rangle$ on a Hilbert space are linear in the first variable and anti-linear in the second.

$\mathcal{S}' := \{\mathrm{A} \in L(\mathcal{H}) | \mathrm{AB} = \mathrm{BA}$ for all $\mathrm{B} \in \mathcal{S}\}$ denotes the *commutant* of a selfadjoint subset $\mathcal{S}$ of $L(\mathcal{H})$, and $\mathcal{S}^-$ is its weak operator closure. A (bounded or unbounded) operator on $\mathcal{H}$ will be called *invertible* if it is injective and has dense range. If it is bounded and its inverse is also bounded we call it *bounded invertible.*

For a von Neumann algebra $\mathcal{M}$ the symbol $\mathcal{P}(\mathcal{M})$ denotes the set of projections in $\mathcal{M}$ and $\mathcal{U}(\mathcal{M})$ denotes the group of unitaries in $\mathcal{M}$. If $\mathcal{M} = L(\mathcal{H})$ we denote $\mathcal{U}(L(\mathcal{H}))$ briefly by $\mathcal{U}(\mathcal{H})$. For a unitary $\mathrm{U} \in L(\mathcal{H})$ we denote by $\mathrm{ad}\,\mathrm{U}$ the implemented isomorphism on $\mathcal{M}$, i. e. $\mathrm{ad}\,\mathrm{U}(\mathrm{M}) := \mathrm{UMU}^*$ for all $\mathrm{M} \in \mathcal{M}$. If $u \in \mathcal{H}$ is a vector in the Hilbert space on which a von Neumann algebra $\mathcal{M}$ acts, we denote the norm-closure of $\mathcal{M}u := \{\mathrm{M}u | \mathrm{M} \in \mathcal{M}\}$ by $[\mathcal{M}u]$. We often do not distinguish between the projection P and the closed subset onto which P maps. Hence, $[\mathcal{M}u]$ is also the projection with image $[\mathcal{M}u]$.

$*$-algebra isomorphisms are always assumed to be $*$-isomorphisms, and convergence of operators is understood in the weak-operator sense if not stated otherwise.

Chapter 2

Tomita-Takesaki Modular Theory

2.1 Introduction to Tomita-Takesaki Modular Theory

In this section we give a short introduction to Tomita-Takesaki modular theory on von Neumann algebras. Our presentation follows [KR86] and [EK98].

The starting point of Tomita-Takesaki modular theory is the following definition:

Definition 2.1.1. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A vector $u \in \mathcal{H}$ is *cyclic* (*generating*) if $\mathcal{M}u$ is dense in $\mathcal{H}$ and it is *separating* if $\mathrm{A}u = 0$ implies $\mathrm{A} = 0$ for $\mathrm{A} \in \mathcal{M}$.

Proposition 2.1.2. *A vector is cyclic for a von Neumann algebra if and only if it is separating for the commutant.*

Let in the following $u \in \mathcal{H}$ be a cyclic and separating vector for the von Neumann algebra $\mathcal{M}$ acting on $\mathcal{H}$. One can then define the following two closable anti-linear operators

$$\begin{aligned} \mathrm{S}_0 : \mathcal{D}(\mathrm{S}_0) = \mathcal{M}u \subset \mathcal{H} &\to \mathcal{H} \\ \mathrm{A}u &\mapsto \mathrm{S}_0\mathrm{A}u = \mathrm{A}^*u \\ \mathrm{F}_0 : \mathcal{D}(\mathrm{F}_0) = \mathcal{M}^{'}u \subset \mathcal{H} &\to \mathcal{H} \\ \mathrm{A}u &\mapsto \mathrm{F}_0\mathrm{A}u = \mathrm{A}^*u. \end{aligned} \tag{2.1.1}$$

Theorem 2.1.3 (Tomita's Theorem). *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector $u \in \mathcal{H}$. Let further S_0, F_0 be the operators defined in (2.1.1) and S, F their closures. Then $\mathrm{S} = \mathrm{F}_0^*$ and $\mathrm{F} = \mathrm{S}_0^*$ hold. Furthermore, $\mathrm{S} = \mathrm{J}\Delta^{1/2}$ is the polar decomposition of S where $\Delta = \mathrm{FS}$ is a positive self-adjoint operator with inverse $\Delta^{-1} = \mathrm{SF}$, and $\mathrm{J}^2 = \mathrm{I}$, $\mathrm{J}\Delta^{it} = \Delta^{it}\mathrm{J}$ $(t \in \mathbb{R})$. Moreover, $\mathrm{J}u = \Delta u = u$, and*

$$\mathrm{J}\mathcal{M}\mathrm{J} = \mathcal{M}^{'}, \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \qquad (t \in \mathbb{R}).$$

We call a von Neumann algebra *standard* if a conjugation J exists such that the mapping $\mathrm{A} \mapsto \mathrm{JA^*J}$ is an anti-isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$ which acts identically on the center. Theorem 2.1.3 then implies that every von Neumann algebra with cyclic and separating vector is standard. Conversely, it can be shown that every standard von Neumann algebra acting on a separable Hilbert space has a cyclic and separating vector (see [SZ79, § 10.15]).

Definition 2.1.4. With the notations of Theorem 2.1.3, we call Δ the *modular operator* and J the *modular conjugation* for $\mathcal{M}$ corresponding to the cyclic and separating vector u. Briefly, the constituents of the pair (Δ, J) are the *modular objects* of $(\mathcal{M}, u)$. We refer to the one-parameter group $\sigma_t = \operatorname{ad} \Delta^{it}$ $(t \in \mathbb{R})$ of automorphisms of $\mathcal{M}$ as the *modular automorphism group.*

In calculations the following corollary is sometimes useful:

Corollary 2.1.5. *Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector $u \in \mathcal{H}$ and corresponding modular objects (Δ, J). If $\pi = \operatorname{ad} \mathrm{U}$ is a unitarily implemented isomorphism from $\mathcal{M}$ onto a von Neumann algebra $\mathcal{N}$, then $\mathrm{U}u$ is cyclic and separating for $\mathcal{N}$ with modular objects $(\mathrm{U}\Delta\mathrm{U}^*, \mathrm{UJU}^*)$.*

The following theorem gives a characterization of the modular automorphism group.

Theorem 2.1.6 (KMS-Condition). *The modular automorphism group σ_t defined in Definition 2.1.4 has the following two properties:*

1. *$\varphi \circ \sigma_t = \varphi$ for all $t \in \mathbb{R}$, where φ is the vector state given by the cyclic and separating vector u.*

2. *For any $\mathrm{A}, \mathrm{B} \in \mathcal{M}$ there exists a bounded continuous function $F(z)$ on $\overline{D}$, where $D = \{z \in \mathbb{C} | 0 < \operatorname{Im} z < 1\}$, such that $F(z)$ is holomorphic on D and*

$$F(t) = \varphi(\sigma_t(\mathrm{A})\mathrm{B}), \quad F(t+i) = \varphi(\mathrm{B}\sigma_t(\mathrm{A})), \quad t \in \mathbb{R}.$$

Conversely, any one-parameter automorphism group σ_t satisfying the above two properties with respect to φ coincides with the modular automorphism group of u.

Remark 2.1.7. Note that property 1 is implied by property 2.

Another concept related to modular theory is the concept of natural cones:

Theorem 2.1.8. *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector $u \in \mathcal{H}$ and corresponding modular objects (Δ, J). Setting*

$$P^\sharp := \overline{\{\mathrm{A}^*\mathrm{A}u | \mathrm{A} \in \mathcal{M}\}}, \quad P^\flat := \mathrm{J}P^\sharp, \quad P^\natural := \overline{\Delta^{1/4}P^\sharp},$$

we get:

1. *The convex cones $P^\sharp$ and $P^\flat$ are mutually dual.*

2. *The cone* $P^\natural$ *is self-dual.*

3. *For* $\mathrm{A} \in \mathcal{M}$*, we have* $\mathrm{AJAJ}P^\natural \subset P^\natural$.

4. *To each positive normal functional* ω *on* $\mathcal{M}$*, there exists a unique* $v \in P^\natural$ *such that* ω *is the vector state* ω_v *associated with* v*. Furthermore,*

$$\|v-w\|^2 \leq \|\omega_v - \omega_w\| \leq \|v-w\| \, \|v+w\| \, ,$$

for $v, w \in P^\natural$.

5. *Let* $v \in P^\natural$*, then* v *is cyclic if and only if it is separating. If* v *is separating then* J *also is the modular conjugation of* v.

Definition 2.1.9. The cone $P^\natural$ defined in the last theorem is called the *natural cone* of the cyclic and separating vector $u \in \mathcal{H}$.

If we have two von Neumann algebras in standard form every isomorphism between them is unitarily implemented. More precisely:

Theorem 2.1.10. *Let* $\mathcal{M}_k$ $(k = 1, 2)$ *be a standard von Neumann algebra acting on a Hilbert space* $\mathcal{H}_k$*. Let further* J_k *be its conjugation and* $P_k^\natural$ *its natural cone. If* $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ *is an isomorphism, then there exists a unitary operator* $\mathrm{U} : \mathcal{H}_1 \to \mathcal{H}_2$ *uniquely determined by the conditions*

1. $\pi(\mathrm{A}) = (\mathrm{ad}\,\mathrm{U})(\mathrm{A})$ *for* $\mathrm{A} \in \mathcal{M}_1$,

2. $\mathrm{J}_2 = (\mathrm{ad}\,\mathrm{U})(\mathrm{J}_1)$,

3. $P_2^\natural = \mathrm{U}(P_1^\natural)$.

This unitary operator U *is called the* standard implementation *of* π.

Consider now the situation of a von Neumann algebra with a normal faithful state ω. The GNS representation with respect to ω provides a cyclic and separating vector u in the representation space such that ω is implemented by u. We call the modular objects corresponding to this vector the *modular objects* of ω. The corresponding modular automorphism group will be denoted by σ_t^ω.

In the following we will present the extension of modular theory to weights. Weights are a generalization of states:

Definition 2.1.11. Let $\mathcal{M}$ be a von Neumann algebra.

1. A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ ($\mathcal{M}^+$ is the set of all positive elements in $\mathcal{M}$) with the properties

$$\begin{aligned} \varphi(\mathrm{A}+\mathrm{B}) &= \varphi(\mathrm{A}) + \varphi(\mathrm{B}) \\ \varphi(\lambda \mathrm{A}) &= \lambda \varphi(\mathrm{A}) \end{aligned} \tag{2.1.2}$$

for all $\mathrm{A}, \mathrm{B} \in \mathcal{M}^+$, $\lambda \in \mathbb{R}^+$ is called a *weight*.

2. A weight φ is called *faithful* if $\varphi(\mathrm{A}^*\mathrm{A}) = 0$ only for $\mathrm{A} = 0$, and it is called *normal* if $\varphi(\sup_i \mathrm{A}_i) = \sup_i \varphi(\mathrm{A}_i)$ for every norm-bounded increasing sequence $(\mathrm{A}_i) \subset \mathcal{M}^+$.

3. For a weight φ let $\mathcal{F}_\varphi$ and $\mathcal{N}_\varphi$ denote the following sets

$$\mathcal{F}_\varphi := \{\mathrm{A} \in \mathcal{M}^+ | \varphi(\mathrm{A}) < \infty\},$$
$$\mathcal{N}_\varphi := \{\mathrm{A} \in \mathcal{M} | \varphi(\mathrm{A}^*\mathrm{A}) < \infty\}.$$

   If the linear hull $\operatorname{lin} \mathcal{F}_\varphi = \mathcal{N}_\varphi^* \mathcal{N}_\varphi$ is σ-weakly dense in $\mathcal{M}$, then φ is called *semifinite*.

4. If a normal semifinite faithful weight φ fulfils $\varphi(\mathrm{A}^*\mathrm{A}) = \varphi(\mathrm{A}\mathrm{A}^*)$ for all $\mathrm{A} \in \mathcal{M}$ then it is calles a *tracial weight*, or, briefly, a *trace*.

We will often abbreviate "normal, semifinite, faithful" to n. s. f. $W_{nsf}(\mathcal{M})$ denotes the set of all n. s. f. weights on $\mathcal{M}$.

Now we can formulate modular theory for n. s. f. weights:

Theorem 2.1.12 (Tomita's Theorem for weights). *Suppose that φ is an n. s. f. weight on a von Neumann algebra $\mathcal{M}$, and $\pi : \mathcal{M} \to L(\mathcal{H}_\varphi)$ is the GNS representation of $\mathcal{M}$ with respect to φ. Then the conjugate linear mapping $\pi(\mathrm{A}) \mapsto \pi(\mathrm{A}^*)$ with domain $\pi(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*)$ is a closable, densely defined operator acting on $\mathcal{H}_\varphi$ and its closure S has polar decomposition $\mathrm{J}\Delta^{1/2}$, where Δ is an invertible positive operator acting on $\mathcal{H}_\varphi$, J is a conjugation acting on $\mathcal{H}_\varphi$, and*

$$\mathrm{J}\pi(\mathcal{M})\mathrm{J} = \pi(\mathcal{M})', \quad \Delta^{it}\pi(\mathcal{M})\Delta^{-it} = \pi(\mathcal{M}) \quad (t \in \mathbb{R}).$$

Definition 2.1.13. We call Δ the *modular operator* and J the *modular conjugation* for $\mathcal{M}$ corresponding to the n. s. f. weight φ. Briefly, the constituents of the pair (Δ, J) are the *modular objects* of $(\mathcal{M}, \varphi)$. We refer to the one-parameter group $\sigma_t = \operatorname{ad} \Delta^{it}$ $(t \in \mathbb{R})$ of automorphisms of $\mathcal{M}$ as the *modular automorphism group*.

Notation. In the setting of Theorem 2.1.12 we denote the GNS Hilbert space $\mathcal{H}_\varphi$ by $L_2(\mathcal{M}, \varphi)$ and call the representation $\pi : \mathcal{M} \to L(L_2(\mathcal{M}, \varphi))$ the *standard representation* of $\mathcal{M}$ (with respect to φ). It can be shown that the standard representation is unique up to unitary equivalence.

Theorem 2.1.14 (KMS-Condition for weights). *The modular automorphism group for $\mathcal{M}$ corresponding to φ is the unique one-parameter group (σ_t) of automorphisms of $\mathcal{M}$ such that*

1. $\varphi(\sigma_t(\mathrm{A})) = \varphi(\mathrm{A})$ *for all* $\mathrm{A} \in \mathcal{M}^+$, $t \in \mathbb{R}$.

2. *For any* $\mathrm{A}, \mathrm{B} \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ *there exists a bounded continuous function $F(z)$ on $\overline{D}$, where $D = \{z \in \mathbb{C} | 0 < \operatorname{Im} z < 1\}$ such that $F(z)$ is holomorphic on D and*

$$F(t) = \varphi(\sigma_t(\mathrm{A})\mathrm{B}), \quad F(t+i) = \varphi(\mathrm{B}\sigma_t(\mathrm{A})), \quad t \in \mathbb{R}.$$

More generally, modular theory can be developed in the context of left Hilbert algebras. A *left Hilbert algebra* is by definition a complex algebra $\mathcal{A}$ with involution $\sharp$ and scalar product $(\cdot|\cdot)$ such that the following holds:

1. multiplication from the left is continuous with respect to the scalar product,
2. $(\xi\eta_1|\eta_2) = (\eta_1|\xi^\sharp\eta_2)$ for any $\xi, \eta_1, \eta_2 \in \mathcal{A}$,
3. $\mathcal{A}^2$ is dense in $\mathcal{A}$,
4. and $\xi \mapsto \xi^\sharp$ $(\xi \in \mathcal{A})$ is a closable anti-linear operator in the Hilbert space $\overline{\mathcal{A}}$.

Left multiplication by elements of $\mathcal{A}$ generate a von Neumann algebra on $\overline{\mathcal{A}}$, the *left von Neumann algebra* of $\mathcal{A}$. Similarly, *right Hilbert algebras* and *right von Neumann algebras* are defined.

In the case of an n. s. f. weight φ on a von Neumann algebra $\mathcal{M}$ the left Hilbert algebra is $\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ and the left von Neumann algebra is $\mathcal{M}$ itself (see e. g. [SZ79]).

For the relation between the modular automorphism groups of two n. s. f. weights we have the following

Theorem 2.1.15 (Connes' Unitary Cocycle Theorem). *Let $\mathcal{M}$ be a von Neumann algebra, and φ, ψ two n. s. f. weights on $\mathcal{M}$. There exists a strong operator continuous mapping $t \mapsto \mathrm{U}_t$ from $\mathbb{R}$ into the unitary group $\mathcal{U}(\mathcal{M})$ of $\mathcal{M}$, such that*

$$\sigma_t^\psi(\mathrm{A}) = \mathrm{U}_t\sigma_t^\varphi(\mathrm{A})\mathrm{U}_t, \quad \mathrm{U}_{s+t} = \mathrm{U}_s\sigma_s^\varphi(\mathrm{U}_t) \quad (s, t \in \mathbb{R}, \quad \mathrm{A} \in \mathcal{M}),$$

where σ^φ and σ^ψ are the modular automorphism groups of φ and ψ, respectively.

If we require that for any $\mathrm{A} \in \mathcal{N}_\varphi \cap \mathcal{N}_\psi^*$ and $\mathrm{B} \in \mathcal{N}_\psi \cap \mathcal{N}_\varphi^*$ there exists a bounded continuous function F on $\overline{D}$, where $D = \{z \in \mathbb{C} | 0 < \mathrm{Im}\, z < 1\}$ such that F is holomorphic on D and

$$F(t) = \psi(\sigma_t^\psi(\mathrm{A})\mathrm{U}_t\mathrm{B}), \quad F(t+i) = \varphi(\mathrm{B}\mathrm{U}_t\sigma_t^\varphi(\mathrm{A})), \quad t \in \mathbb{R},$$

then U_t is unique, it is called Connes' cocycle, *and is denoted by $[D\psi : D\varphi]_t$.*

The following Radon-Nikodym type theorem by Pedersen and Takesaki states how the Connes' Cocycle can be computed explicitly in some cases:

Theorem 2.1.16. *Let φ, ψ be two n. s. f. weights on the von Neumann algebra $\mathcal{M}$. Then the following conditions are equivalent:*

1. *$\psi \circ \sigma_t^\varphi = \psi$ for all $t \in \mathbb{R}$.*
2. *$[D\psi : D\varphi]_t \in \mathcal{M}^\psi$ for all $t \in \mathbb{R}$ where*

$$\mathcal{M}^\psi := \{\mathrm{A} \in \mathcal{M} | \sigma_t^\psi(\mathrm{A}) = \mathrm{A} \textit{ for all } t \in \mathbb{R}\}$$

   is the centralizer *of ψ.*

3. *$[D\psi : D\varphi]_t \in \mathcal{M}^\varphi$ for all $t \in \mathbb{R}$.*
4. *$[D\psi : D\varphi]_t \in \mathcal{M}^\varphi$ is a so-continuous group of unitary elements of $\mathcal{M}$.*

5. *There exists a positive invertible operator* A *affiliated with* $\mathcal{M}$ *such that* $\psi = \varphi_{\mathrm{A}}$ *where* $\varphi_{\mathrm{A}} := \varphi(\mathrm{A}^{1/2} \cdot \mathrm{A}^{1/2})$*. (We call a closed operator* A affiliated with $\mathcal{M}$ *and write*
$$\mathrm{A}\eta\mathcal{M}$$
if A *commutes with all unitaries from the commutant* $\mathcal{M}'$ *of* $\mathcal{M}$ *.)*

6. $\varphi \circ \sigma_t^{\psi} = \varphi$ *for all* $t \in \mathbb{R}$*.*

Since the modular automorphism group corresponding to a trace tr is trivial, $\sigma_t^{\mathrm{tr}} = \mathrm{id}$ for all $t \in \mathbb{R}$, we get the following corollary for semifinite von Neumann algebras (a *semifinite von Neumann algebra* is an algebra which is not of type III and therefore possesses a trace). This implies that the modular automorphism group is inner in this case:

Corollary 2.1.17. *Let* $\mathcal{M}$ *be a semifinite von Neumann algebra with an n. s. f. trace* tr *and let* φ *be an n. s. f. weight on* $\mathcal{M}$*. Then there exists a positive invertible operator* $\mathrm{H}\eta\mathcal{M}$ *such that* $\varphi = \mathrm{tr}_{\mathrm{H}}$ *and*
$$\sigma_t^{\varphi} = \mathrm{ad}\,\mathrm{H}^{it}.$$

Remark 2.1.18. We will show later (Theorem 3.3.9) that if the semifinite von Neumann algebra is in standard form the modular unitary group has the form
$$\Delta^{it} = \mathrm{H}^{it}\mathrm{J}\mathrm{H}^{it}\mathrm{J}.$$

2.2 Constructions with von Neumann Algebras

In this section we present some useful constructions with von Neumann algebras, give examples of them and show how the modular objects can be obtained in these cases.

2.2.1 Tensor Product of von Neumann Algebras

We again follow the presentation of Kadison and Ringrose [KR86]. Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$, respectively. We denote by $\mathcal{M}_0$ the $*$-algebra acting on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ that consists of all finite sums of operators of the form $\mathrm{A}_1 \otimes \cdots \otimes \mathrm{A}_{\mathrm{n}}$ ($\mathrm{A}_j \in \mathcal{M}_j$ for $j = 1, \ldots, n$). The *von Neumann algebra tensor product* $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ is defined to be the von Neumann algebra $\mathcal{M}_0^- = (\mathcal{M}_0)''$ generated by $\mathcal{M}_0$.

Theorem 2.2.1. *Let* $\mathcal{M}_1, \ldots \mathcal{M}_n$ *be von Neumann algebras.*

1. *If* $\mathcal{M}_1, \ldots \mathcal{M}_n$ *are generated by subsets* $\mathcal{S}_1, \ldots, \mathcal{S}_n$*, respectively, then* $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ *is generated by the set*
$$\mathcal{S} = \{\mathrm{S}_1 \otimes \cdots \otimes \mathrm{S}_n | \mathrm{S}_1 \in \mathcal{S}_1, \ldots, \mathrm{S}_n \in \mathcal{S}_n\}.$$

2. *The tensor product is associative.*

3. *If ω_j is a normal state of $\mathcal{M}_j$ ($j = 1, \ldots, n$), then there is a unique normal state ω of $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ such that*

$$\omega(\mathrm{A}_1 \otimes \cdots \otimes \mathrm{A}_n) = \omega_1(\mathrm{A}_1) \cdots \omega_n(\mathrm{A}_n)$$

for $\mathrm{A}_1 \in \mathcal{M}_1, \ldots, \mathrm{A}_n \in \mathcal{M}_n$. Such a state is called a normal product state.

4. *The predual of $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ is the norm-closed subspace of the dual space of $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ generated by the set of all normal product states.*

5. *Let α_j be an isomorphism from $\mathcal{M}_j$ onto a von Neumann algebra $\mathcal{N}_j$ ($j = 1, \ldots, n$). Then there is a unique isomorphism $\alpha =: \alpha_1 \otimes \cdots \otimes \alpha_n$ from $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ onto $\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n$ such that*

$$\alpha(\mathrm{A}_1 \otimes \cdots \otimes \mathrm{A}_n) = \alpha_1(\mathrm{A}_1) \otimes \cdots \otimes \alpha(\mathrm{A}_n)$$

for $\mathrm{A}_1 \in \mathcal{M}_1, \ldots, \mathrm{A}_n \in \mathcal{M}_n$.

6. *The commutant of $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ is $\mathcal{M}_1^{'} \otimes \cdots \otimes \mathcal{M}_n^{'}$.*

For the type of the tensor product see Table 2.1, cf. [KR86, Table 11.1].

Table 2.1: The type of $\mathcal{M} \otimes \mathcal{N}$

		type of $\mathcal{N}$				
		I_n	I_∞	II_1	II_∞	III
type of $\mathcal{M}$	I_m	I_{mn}	I_∞	II_1	II_∞	III
	I_∞	I_∞	I_∞	II_∞	II_∞	III
	II_1	II_1	II_∞	II_1	II_∞	III
	II_∞	II_∞	II_∞	II_∞	II_∞	III
	III	III	III	III	III	III

Remark 2.2.2. The table shows that the tensor product of a type II_1 algebra with a type I_∞ algebra is always of type II_∞. The converse is also true, i. e. every type II_∞ algebra is (isomorphic to) the tensor product of a type II_1 and a type I_∞ algebra (see e. g. [KR86, Theorem 6.7.10] and Theorem 3.3.2).

Let now $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ with cyclic and separating vectors $u_1 \in \mathcal{H}_1, \ldots, u_n \in \mathcal{H}_n$, respectively. Then $u := u_1 \otimes \cdots \otimes u_n \in \mathcal{H} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is a cyclic and separating vector for $\mathcal{M} := \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ acting on $\mathcal{H}$. For the modular objects we have the following

Theorem 2.2.3. *The modular objects (Δ, J) of $(\mathcal{M}, u)$ are*

$$\Delta = \Delta_1 \otimes \cdots \otimes \Delta_n \quad \text{and} \quad \mathrm{J} = \mathrm{J}_1 \otimes \cdots \otimes \mathrm{J}_n$$

where (Δ_k, J_k) are the modular objects of $(\mathcal{M}_k, u_k)$ ($k = 1, \ldots, n$).

2.2.2 (Discrete) Crossed Products

For details of the following construction we refer to [KR86] and [Str81]. Suppose that $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and G is a discrete group (with unit e). An *action (automorphic representation)* of G on $\mathcal{M}$ is a homomorphism $\alpha : g \mapsto \alpha_g$ from G into the group of automorphisms of $\mathcal{M}$. Such an action is *unitarily implemented* if there is a unitary representation $g \mapsto \mathrm{U}(g)$ of G on $\mathcal{H}$ such that $\alpha_g = \operatorname{ad} \mathrm{U}_g$. Furthermore, α is *properly outer* if the following condition is satisfied: if $\mathrm{A} \in \mathcal{M}$ and $g \in G$ such that $\mathrm{BA} = \mathrm{A}\alpha_g(\mathrm{B})$ for all $\mathrm{B} \in \mathcal{M}$, then $g = e$ or $\mathrm{A} = 0$. We say α *acts freely* on $\mathcal{M}$ if the following condition is satisfied: if $\mathrm{A} \in \mathcal{M}$ and $g \in G$ such that $\mathrm{CA} = \mathrm{A}\alpha_g(\mathrm{C})$ for all $\mathrm{C} \in \mathcal{M}$, then $g = e$ or $\mathrm{A} = 0$. Finally, the action α is *ergodic* if $\mathcal{M}^\alpha = \mathbb{C}$, where $\mathcal{M}^\alpha := \{\mathrm{A} \in \mathcal{M} | \alpha_g(\mathrm{A}) = \mathrm{A} \text{ for all } g \in G\}$ is the *centralizer* (or *fixed point algebra*) of the action α.

Definition 2.2.4. The *(implemented) crossed product* of $\mathcal{M}$ by α is the von Neumann algebra $\mathcal{R}(\mathcal{M}, \alpha) = \mathcal{M} \rtimes_\alpha G$ acting on $\mathcal{H} \otimes l_2(G)$ which is generated by

$$\Phi(\mathrm{A}) := \mathrm{A} \otimes \mathrm{I}, \quad \mathrm{V}(g) := \mathrm{U}(g) \otimes l_g, \quad (\mathrm{A} \in \mathcal{M}, g \in G),$$

where $(l_g y)(h) := y(g^{-1}h)$ for $y \in l_2(G)$ and $g, h \in G$.

Writing operators on $\mathcal{H} \otimes l_2(G)$ as matrices $[\mathrm{S}(p,q)]$, where p and q take values in G and $\mathrm{S}(p,q) \in L(\mathcal{H})$, we get $\Phi(\mathrm{A}) = [\delta_{p,q}\mathrm{A}]$ and $\mathrm{V}(g) = [\delta_{p,gq}\mathrm{U}(g)]$. Elementary matrix calculations show that $\mathcal{R}(\mathcal{M}, \alpha)$ consists of all elements of $L(\mathcal{H}) \otimes l_2(G)$ which have matrices of the form $[\mathrm{U}(pq^{-1})\mathrm{A}(pq^{-1})]$, for some mapping $g \mapsto \mathrm{A}(g) : G \to \mathcal{M}$, while the commutant $\mathcal{R}(\mathcal{M}, \alpha)'$ consists of all operators which have matrices of the form $[\mathrm{U}(p)\mathrm{A}'(q^{-1}p)\mathrm{U}(p)^*]$, for some mapping $g \mapsto \mathrm{A}'(g) : G \to \mathcal{M}'$.

Theorem 2.2.5. *Let α be an action of the discrete group G on the von Neumann algebra $\mathcal{M}$.*

1. *(Relative Commutant Theorem)*
   (a) *α is properly outer if and only if $\Phi(\mathcal{M}') \cap \mathcal{R}(\mathcal{M}, \alpha) = \Phi(\mathcal{Z}(\mathcal{M}))$ where $\Phi(\mathrm{A}') := \mathrm{A}' \otimes \mathrm{I}$ for $\mathrm{A}' \in \mathcal{M}'$.*
   (b) *α acts freely on $\mathcal{Z}(\mathcal{M})$ if and only if $\Phi(\mathcal{Z}(\mathcal{M}))' \cap \mathcal{R}(\mathcal{M}, \alpha) = \Phi(\mathcal{M})$.*
2. *If α is properly outer and its restriction to $\mathcal{Z}(\mathcal{M})$ is ergodic then the crossed product $\mathcal{R}(\mathcal{M}, \alpha)$ is a factor.*
3. *The crossed product $\mathcal{R}(\mathcal{M}, \alpha)$ is a finite von Neumann algebra if and only if there exists an α-invariant finite normal faithful trace on $\mathcal{M}$.*
4. *If there exists an α-invariant n. s. f. trace on $\mathcal{M}$ then the crossed product $\mathcal{R}(\mathcal{M}, \alpha)$ is semifinite.*

In the following we will give two important examples for the crossed product construction.

Example 2.2.6. Let $\mathcal{H}$ be the Hilbert space $L_2([0,1),\lambda)$ of square-integrable functions with respect to Lebesgue measure λ on the Borel σ-algebra of the unit interval. Let further $\mathcal{A} = L_\infty([0,1),\lambda)$ be the maximal abelian von Neumann algebra of bounded measurable functions acting on the Hilbert space $\mathcal{H}$ as multiplication operators. Moreover, let G be the group of all rational translations, modulo 1, of $[0,1)$. Then we can define an action α of G on $\mathcal{A}$ such that $\alpha_g(f)(x) = f(\{x-g\})$ for all $g \in G$ $(x \in [0,1))$ where $\{a\}$ denotes the fractional part of the real number a. This action acts freely on $\mathcal{A}$ (hence, it is also properly outer). It is also ergodic such that $\mathcal{R}(\mathcal{A},\alpha)$ is a factor. $\mathcal{R}(\mathcal{A},\alpha)$ is finite since the integral with respect to λ is a normal faithful α-invariant finite trace on $\mathcal{A}$, and it is of type II because $\mathcal{H} \otimes l_2(G)$ is infinite dimensional. Together with Theorem 2.2.5 these properties imply that we have constructed a type II_1 factor.

Example 2.2.7. Let now $\mathcal{K}$ be the Hilbert space $L_2(\mathbb{R},\lambda))$ of square-integrable functions with respect to Lebesgue measure λ on the Borel σ-algebra of the the real line $\mathbb{R}$. Let further $\mathcal{B} = L_\infty(\mathbb{R},\lambda)$ be the maximal abelian von Neumann algebra of bounded measurable functions acting on the Hilbert space $\mathcal{K}$ as multiplication operators. Moreover, let F be the group of all rational translations of $\mathbb{R}$. Then we can define an action of F on $\mathcal{B}$ such that $\beta_g(f)(x) = f(x-g)$ for all $g \in F$ $(x \in \mathbb{R})$. This action acts freely on $\mathcal{B}$ (hence, it is also properly outer). It is also ergodic such that $\mathcal{R}(\mathcal{B},\beta)$ is a factor. $\mathcal{R}(\mathcal{B},\beta)$ is semifinite since the integral with respect to λ is a n. s. f. β-invariant trace on $\mathcal{B}$. Since the measure λ is infinite this trace is also infinite. Because of the uniqueness of the trace on a factor $\mathcal{R}(\mathcal{B},\beta)$ is infinite as well. $\mathcal{R}(\mathcal{B},\beta)$ can not be of type I because the trace on a I_∞ factor restricted to the set of projections admits only discrete values whereas Lebesgue measure on $\mathbb{R}$ admits a continuum of values. Together with Theorem 2.2.5 these properties imply that we have constructed a type II_∞ factor.

If φ is an n. s. f. weight on the von Neumann algebra $\mathcal{M}$ then we can define a n. s. f. weight on $\mathcal{R}(\mathcal{M},\alpha)$ by setting $\hat{\varphi}(\mathrm{A}) := \varphi(\mathrm{A}(e))$ $(\mathrm{A} = [\mathrm{U}(pq^{-1})\mathrm{A}(pq^{-1})] \in \mathcal{R}(\mathcal{M},\alpha)^+)$. It is called the *dual weight* of φ.

Theorem 2.2.8. *Adopt the above notations. Let $\sigma_t^{\hat{\varphi}}$ be the modular automorphism group of $\hat{\varphi}$ and let $\mathrm{A} = [\mathrm{U}(pq^{-1})\mathrm{A}(pq^{-1})] \in \mathcal{R}(\mathcal{M},\alpha)$. Then*

$$\sigma_t^{\hat{\varphi}}(\mathrm{A}) = [\mathrm{U}(pq^{-1})\mathrm{B}_t(pq^{-1})] \quad (t \in \mathbb{R})$$

where $\mathrm{B}_t(p) := [D(\varphi \circ \alpha_p) : D\varphi]_t \sigma_t^{\varphi}(\mathrm{A}(p))$ $(p \in G,\ t \in \mathbb{R})$.

Remark 2.2.9. If $\mathcal{M}$ is a von Neumann algebra with cyclic and separating vector $u \in \mathcal{H}$ the crossed product $\mathcal{R}(\mathcal{M},\alpha)$ has the cyclic and separating vector $[\delta_{p,e}u]_{p\in G} \in \mathcal{H} \otimes l_2(G)$, the so-called *dual vector* of u. We will show later (Theorem 3.4.6) that the modular group Δ^{it} for $\mathcal{R}(\mathcal{M},\alpha)$ corresponding to $[\delta_{p,e}u]_p$ has the form

$$\Delta^{it} = \Phi(\mathrm{H})^{it}\mathrm{J}\Phi(\mathrm{H})^{it}\mathrm{J}\Delta_a^{it},$$

if $\mathcal{M}$ is semifinite with n. s. f. trace tr and Δ_a is the positive operator with matrix $[\delta_{p,q}\mathrm{U}(p)\mathrm{A}_p\mathrm{U}(p)^*]$ and $\mathrm{tr} \circ \alpha_p = \mathrm{tr}_{\mathrm{A}_p}$.

2.3 Mathematical Applications of Modular Theory

Beside the fact that modular theory furnishes the standard form of every von Neumann algebra acting on a separable Hilbert space, there are much more applications. In this section we present some examples. The choice is mainly motivated by the need of these example for the further development of this thesis. We present the Connes' classification theory of type III factors [Con73] and Haagerup's theory of operator valued weights [Haa79b, Haa79c]. For further applications we refer to [Str81], for instance. Recently modular theory, especially the modular conjugation, has become important in the index theory of subfactors, first introduced by Jones [Jon83] or, more generally, of arbitrary algebra inclusions defined by Kosaki [Kos86], and in the related basic construction (see also [Jon91]).

Our presentation of the classification theory of type III factors and the theory of operator valued weights follows [Str81]. Let in the following $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ with an n. s. f. weight φ. As in §2.1 we denote the modular automorphism group corresponding to φ by σ_t^φ. Then we can define an invariant of $\mathcal{M}$ by

$$T(\mathcal{M}) := \{t \in \mathbb{R} | \sigma_t^\varphi \text{ is an inner automorphism}\}.$$

Because of the Unitary Cocycle Theorem (see Theorem 2.1.15) this is an invariant which is independent of the weight φ. The group property of σ_t^φ implies that $T(\mathcal{M})$ is a subgroup of the additive group $\mathbb{R}$. Furthermore, Corollary 2.1.17 states that $T(\mathcal{M}) = \mathbb{R}$ for all semifinite algebras. In the separable case the converse also is true, i. e. if $T(\mathcal{M}) = \mathbb{R}$ the algebra $\mathcal{M}$ is semifinite (see e. g. [Str81, Proposition 27.2]). Hence, the invariant $T(\mathcal{M})$ distinguishes between semifinite algebras and type III algebras but not between the various semifinite ones.

In the following we restrict our considerations to the case of factors. We consider the following invariant:

$$S(\mathcal{M}) := \bigcap_{\varphi \in W_{nsf}(\mathcal{M})} \sigma(\Delta_\varphi)$$

where $\sigma(\Delta_\varphi)$ denotes the spectrum of the (positive) modular operator Δ_φ corresponding to φ. This invariant has the following properties:

Theorem 2.3.1. *Adopt the above notations.*

1. *$\mathcal{M}$ is semifinite if and only if $0 \notin S(\mathcal{M})$. In this case, $S(\mathcal{M}) = \{1\}$.*
2. *$S(\mathcal{M}) \cap \mathbb{R}_*^+$ is a closed subgroup of the multiplicative group $\mathbb{R}_*^+$.*
3. *If $\mathcal{M}$ is not both infinite and semifinite then*

$$S(\mathcal{M}) := \bigcap_{\varphi} \sigma(\Delta_\varphi), \tag{2.3.1}$$

where the intersection is taken over all faithful, normal states φ on $\mathcal{M}$. If $\mathcal{M}$ is both infinite and semifinite, i. e. $\mathcal{M}$ is of type I_∞ or type II_∞, $S(\mathcal{M}) = \{1\}$, while the right hand side of (2.3.1) is $\{0, 1\}$.

4. Let $S(\mathcal{M}) \neq \{0,1\}$. Then $T(\mathcal{M})$ is the annihilator of $S(\mathcal{M}) \cap \mathbb{R}_*^+$ in $\mathbb{R}$.

Let now $\mathcal{M}$ be a type III factor. Since $S(\mathcal{M}) \cap \mathbb{R}_*^+$ is a closed subgroup of $\mathbb{R}_*^+$ and $S(\mathcal{M})$ is a closed subset of $[0,\infty)$ we have the following three possibilities for $S(\mathcal{M})$:

III_0. $S(\mathcal{M}) = \{0,1\}$,

III_1. $S(\mathcal{M}) = [0,\infty)$,

III_λ. $S(\mathcal{M}) = \{0\} \cup \{\lambda^n | n \in \mathbb{Z}\}$ for $0 < \lambda < 1$.

Definition 2.3.2. Let $\mathcal{M}$ be a factor of type III. Then $\mathcal{M}$ is *of type III_0*, *of type III_1*, or *of type III_λ* for $0 < \lambda < 1$, if its invariant $S(\mathcal{M})$ is of the form III_0, III_1, or III_λ, respectively.

Remark 2.3.3. For the invariant $T(\mathcal{M})$ we have the following possibilities in the above cases:

III_0. There are examples of factors $\mathcal{M}$ with $T(\mathcal{M}) = \{nt | n \in \mathbb{Z}\}$ for every $t \in \mathbb{R}$ (see [Con73]).

III_1. $T(\mathcal{M}) = \{0\}$.

III_λ. $T(\mathcal{M}) = \{2\pi n/\ln(\lambda) | n \in \mathbb{Z}\}$.

Note that the invariant $T(\mathcal{M})$ does not distinguish between type III_0 and the other types, since Connes' example with $t = 0$ ($t = 2\pi/\ln(\lambda)$) gives a type III_0 factor with $T(\mathcal{M}) = \{0\}$ ($T(\mathcal{M}) = \{2\pi n/\ln(\lambda) | n \in \mathbb{Z}\}$) as in the type III_1 (type III_λ) case.

With the help of the crossed product construction of §2.2 we can construct examples of type III_λ factors:

Proposition 2.3.4. *Let $\mathcal{N}$ be a factor of type II_∞, tr be an n. s. f. trace on $\mathcal{N}$, and $\theta \in \mathrm{aut}(\mathcal{N})$ be an automorphism such that $\mathrm{tr} \circ \theta = \lambda \, \mathrm{tr}$ for $0 < \lambda < 1$. Then the action $\theta : \mathbb{Z} \ni \mapsto \theta^n \in \mathrm{aut}\, \mathcal{N}$ is properly outer and the crossed product $\mathcal{M} = \mathcal{R}(\mathcal{N}, \theta)$ is a factor of type III_λ.*

The dual weight τ of the trace tr in this construction has the following properties:

- $\sigma_\tau^T = \mathrm{id}$ for $T := -\frac{2\pi}{\ln \lambda}$.
- $\tau(\mathrm{I}) = \infty$.

Such n. s. f. weights are called *generalized traces* or *λ-traces.*

A similar construction leads to factors of type III_0:

Proposition 2.3.5. *Let $\mathcal{N}$ be a von Neumann algebra of type II_∞ with diffuse center (i. e. the center has no minimal projections), let tr be an n. s. f. trace on $\mathcal{N}$, and let $\theta \in \mathrm{aut}(\mathcal{N})$ be an automorphism acting ergodically on $\mathcal{Z}(\mathcal{N})$ such that $\mathrm{tr} \circ \theta \leq \lambda_0 \, \mathrm{tr}$ for one $0 < \lambda_0 < 1$. Then the action $\theta : \mathbb{Z} \ni \mapsto \theta^n \in \mathrm{aut}\, \mathcal{N}$ is free on the center of $\mathcal{N}$ and the crossed product $\mathcal{M} = \mathcal{R}(\mathcal{N}, \theta)$ is a factor of type III_0.*

The dual weight τ of the trace tr in this construction has the following properties:

1. 1 is an isolated point in the spectrum of Δ_τ.
2. The centralizer $\mathcal{M}^\tau$ is properly infinite.

N. s. f. weights fulfilling property 1 are called *lacunary weights* and n. s. f. weights fulfilling property 2 are called weights of *infinite multiplicity.* Hence, τ is a lacunary weight of infinite multiplicity.

The triples $(\mathcal{N}, \theta, \mathrm{tr})$ of Proposition 2.3.4 and Proposition 2.3.5 are referred to as a *discrete decomposition of type* III_λ and a *discrete decomposition of type* III_0, respectively.

Conversely, it can be shown that every factor of type III_0 or type III_λ can be constructed in this way:

Theorem 2.3.6. *Let $\mathcal{M}$ be a factor of type III_λ ($0 \leq \lambda < 1$). Then there exists a discrete decomposition $(\mathcal{N}, \theta, \mathrm{tr})$ of type III_λ such that $\mathcal{M}$ is isomorphic to $\mathcal{R}(\mathcal{N}, \theta)$.*

Remark 2.3.7.

1. In the proof of this theorem the existence of a generalized trace (a lacunary weight of infinite multiplicity) τ is crucial. The algebra $\mathcal{N}$ can be chosen as the centralizer of τ.
2. In the type III_1 case such a discrete decomposition in general does not exist (see [Con74]). However, if we also consider *continuous* crossed products, the crossed product $\mathcal{R}(\mathcal{M}, \sigma)$ of a von Neumann algebra $\mathcal{M}$ with its modular automorphism group σ is always a semifinite algebra (a type II_∞ one, if $\mathcal{M}$ is of type III) and there is an action $\hat{\sigma}$ of $\mathbb{R}$ (the dual action) on the crossed product such that $\mathcal{M}$ is isomorphic to the crossed product $\mathcal{R}(\mathcal{R}(\mathcal{M}, \sigma), \hat{\sigma})$.

   We will show later that in special cases there also exist discrete decompositions of type III_1 factors (see §3.4.2).

Remark 2.3.8. In §2.2 we remarked that every II_∞ algebra is the tensor product of a type II_1 algebra with the I_∞ factor. We have now seen that each type III_λ factor ($0 \leq \lambda < 1$) is the crossed product of a type II_∞ algebra with a discrete action. This means that somehow all von Neumann algebras are reduced to the type II_1 and type I case.

A further application of modular theory is the following existence and uniqueness result for operator valued weights. An *operator valued weight* on a von Neumann algebra $\mathcal{M}$ with values in a subalgebra $\mathcal{N} \subset \mathcal{M}$ is a mapping $\mathrm{E} : \mathcal{M}^+ \to \overline{\mathcal{N}^+}$ with the properties

$$\begin{aligned}
\mathrm{E}(\mathrm{A}+\mathrm{B}) &= \mathrm{E}(\mathrm{A}) + \mathrm{E}(\mathrm{B}), \\
\mathrm{E}(\lambda \mathrm{A}) &= \lambda \mathrm{E}(\mathrm{A}), \\
\mathrm{E}(\mathrm{N}^* \mathrm{A} \mathrm{N}) &= \mathrm{N}^* \mathrm{E}(\mathrm{A}) \mathrm{N},
\end{aligned}$$

where $\mathrm{A}, \mathrm{B} \in \mathcal{M}^+$, $\lambda \geq 0$, $\mathrm{N} \in \mathcal{N}$. $\overline{\mathcal{N}^+}$ is the *extended positive part* of the von Neumann algebra $\mathcal{N}$ which is the set of all functions $m : \mathcal{N}_*^+ \to [0, \infty]$ such that:

$$m(\varphi + \psi) = m(\varphi) + m(\psi),$$
$$m(\lambda\varphi) = \lambda m(\varphi),$$
$$m \text{ is lower semicontinuous,}$$

with $\varphi, \psi \in \mathcal{N}_*^+$, $\lambda \geq 0$. Note that every positive operator affiliated with $\mathcal{N}$ can be considered as an element of $\overline{\mathcal{N}^+}$ [Str81, § 11.2]. We can define the notions of semifiniteness, normality and faithfulness similarly to the corresponding notions for (complex valued) weights. An operator valued weight E with $\mathrm{E}(\mathrm{I}) = \mathrm{I}$ is called *conditional expectation*. An *operator valued trace* is an operator valued weight with $\mathrm{E}(\mathrm{A}^*\mathrm{A}) = \mathrm{E}(\mathrm{A}\mathrm{A}^*)$ for all $\mathrm{A} \in \mathcal{M}$.

Remark 2.3.9. Note that every n. s. f. weight is an n. s. f. operator valued weight with values in $\mathbb{C}$ and that every normal operator valued weight can be extended uniquely to the extended positive part of $\mathcal{M}$ [Str81, Proposition 11.4].

Theorem 2.3.10. *Let φ be an n. s. f. weight on the von Neumann algebra $\mathcal{M}$ and ψ an n. s. f. weight on the subalgebra $\mathcal{N} \subset \mathcal{M}$. If*

$$\sigma_t^\varphi(\mathrm{N}) = \sigma_t^\psi(\mathrm{N})$$

for all $\mathrm{N} \in \mathcal{N}$ and $t \in \mathbb{R}$, there exists a unique n. s. f. operator valued weight $\mathrm{E} : \mathcal{M}^+ \to \overline{\mathcal{N}^+}$ such that $\varphi = \psi \circ \mathrm{E}$.

Remark 2.3.11.

1. Let $\mathcal{M}$ be a semifinite algebra with trace tr, and let ω be an n. s. f. weight on the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$. Then Theorem 2.3.10 implies the existence of an n. s. f. operator valued weight $\mathrm{tr}_\mathcal{M}$ with values in the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ such that $\mathrm{tr}(\mathrm{A}) = \omega(\mathrm{tr}_\mathcal{M}(\mathrm{A}))$ for $\mathrm{A} \in \mathcal{M}^+$. $\mathrm{tr}_\mathcal{M}$ is an operator valued trace, a so-called *central trace*.

2. Central traces are uniquely determined up to a positive element affiliated with the center of $\mathcal{M}$. If $\mathcal{M}$ is a type I algebra, i. e. if there is an abelian projection E with central carrier I, then $\mathrm{tr}_\mathcal{M}$ is unique as soon as we demand $\mathrm{tr}_\mathcal{M}(\mathrm{E}) = \mathrm{I}$. If $\mathcal{M}$ is finite and we require additionally $\mathrm{tr}(\mathrm{I}) = \omega(\mathrm{I}) = 1$, hence $\mathrm{tr}_\mathcal{M}(\mathrm{I}) = \mathrm{I}$, then $\mathrm{tr}_\mathcal{M}$ is unique as well. In both cases it is called the *canonical central trace*.

3. Two projections $\mathrm{P}, \mathrm{Q} \in \mathcal{P}(\mathcal{M})$ are equivalent if and only if

$$\mathrm{tr}_\mathcal{M}(\mathrm{P}) = \mathrm{tr}_\mathcal{M}(\mathrm{Q}). \tag{2.3.2}$$

2.4 Physical Applications of Modular Theory

One of the main motivations for research in modular theory nowadays is its application in mathematical physics, especially in algebraic quantum field theory. In this section we present some important results which are part of the initial motivation for this thesis.

The first observation is an application in quantum statistical mechanics which motivated the term "KMS-Condition" for Theorem 2.1.6 and Theorem 2.1.14 (for the following see e. g. [Haa92]). In (non-relativistic) quantum mechanics the dynamics of a 1-component system confined to a box with finite volume is given by the *Hamilton operator* H on a suitable Hilbert space $\mathcal{H}$. A general state of the system is given by a *density matrix* ρ (a positive trace class operator). The expectation value of an observable $\mathrm{A} \in L(\mathcal{H})$ is then given by

$$\omega(\mathrm{A}) = \mathrm{tr}(\rho \mathrm{A}).$$

Furthermore, the *Gibbs states* are the equilibrium states of the canonical ensemble at inverse temperature $\beta = (kT)^{-1}$ (k is the Boltzmann constant, T the absolute temperature) defined by

$$\rho_\beta = Z^{-1} e^{-\beta \mathrm{H}},$$

where $Z = \mathrm{tr}(e^{-\beta \mathrm{H}})$ is the partition function. These states have the following property which was first pointed out by Kubo and was later used by Martin and Schwinger to define "thermodynamic Green's functions":

$$G^{(\beta)}_{\mathrm{A,B}}(\tau) = F^{(\beta)}_{\mathrm{A,B}}(\tau - i\beta), \tag{2.4.1}$$

where

$$F^{(\beta)}_{\mathrm{A,B}}(z) = \omega_\beta(\alpha_z(\mathrm{A})\mathrm{B})$$
$$G^{(\beta)}_{\mathrm{A,B}}(z) = \omega_\beta(\mathrm{B}\alpha_z(\mathrm{A}))$$

and $\alpha_z(\mathrm{A}) = e^{i\mathrm{H}z}\mathrm{A}e^{-i\mathrm{H}z}$ is the (complex) time evolution of an observable $\mathrm{A} \in L(\mathcal{H})$. This observation motivated Haag, Hugenholtz, and Winnink ([HHW67]) to postulate (2.4.1) together with an analycity requirement for F as the defining property of equilibrium states in the general algebraic setting of quantum statistical mechanics. They called it *KMS-condition.* Comparison with Theorem 2.1.6 implies that this is exactly the KMS-condition of the modular automorphism group of a faithful normal state if we set $t = -\tau/\beta$. Thus, the equilibrium states with inverse temperature β are the faithful normal states whose modular automorphism group σ_t is the time translation group. Further details of this application can be found in [BR81].

The next application illuminates the significance of modular theory in the theory of local observables in the sense of Araki, Haag, and Kastler (for the following see e. g. [Bor00]). We consider the case of the so-called *vacuum representation* here, i. e. we associate to every bounded open region $\mathcal{O}$ in Minkowski space $\mathbb{R}^d$ a von Neumann algebra $\mathcal{M}(\mathcal{O}) \subset L(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$ with the following properties:

1. If $\mathcal{O}_1 \subset \mathcal{O}_2$ are bounded open regions, then $\mathcal{M}(\mathcal{O}_1) \subset \mathcal{M}(\mathcal{O}_2)$ (*isotony*).

2. If $\mathcal{O}_1, \mathcal{O}_2$ are space-like separated regions, then the corresponding algebras commute, i. e. $\mathcal{M}(\mathcal{O}_1) \subset \mathcal{M}(\mathcal{O}_2)'$ (*locality*).

3. There exists a strongly continuous unitary representation $\mathrm{U}(\Lambda, a)$ of the Poincaré group on $\mathcal{H}$, such that
   (a) The spectrum of the translations is contained in the forward light cone.
   (b) There exists a (unique) vector $\Omega \in \mathcal{H}$ (the *vacuum*) such that $\mathrm{U}(\Lambda, a)\Omega = \Omega$.
   (c) $\operatorname{ad} \mathrm{U}(\Lambda, a)\mathcal{M}(\mathcal{O}) = \mathcal{M}(\Lambda\mathcal{O} + a)$ (*covariance*).
4. The vacuum is cyclic for the *quasi-local algebra* $\mathcal{A} := C^*(\bigcup_{\mathcal{O}} \mathcal{M}(\mathcal{O}))$.
5. The global algebra $\mathcal{M} := \mathcal{A}''$ is a non-trivial factor ($\mathcal{M} \neq \mathbb{C}$).
6. We set $\mathcal{M}(\mathcal{U}) := (\bigcup_{\mathcal{O}\subset\mathcal{U}} \mathcal{M}(\mathcal{O}))''$ for unbounded open regions $\mathcal{U}$. Then $(\bigcup_{a\in\mathbb{R}^d} \mathcal{M}(\mathcal{O} + a))'' = \mathcal{M}(\mathbb{R}^d)$ for every $\mathcal{O}$ (*weak additivity*).

There are more general axiom system, the axioms stated here are sufficient for our purposes, however. For more details we refer to [Haa92] or [BW92], for instance. In this framework we have

Theorem 2.4.1 (Reeh-Schlieder-Theorem). *The vacuum Ω is cyclic for $\mathcal{M}(\mathcal{O})$ for any non-void open region $\mathcal{O}$.*

This theorem and Proposition 2.1.2 together with the axiom of locality then implies that modular theory can be applied to the local algebra $\mathcal{M}(\mathcal{O})$ if $\mathcal{O}$ has a non-void causal complement.

In some circumstances the modular objects have geometrical meanings. Let, e. g. , $\mathcal{W}$ be the following wedge in Minkowski space $\mathbb{R}^4$

$$\mathcal{W} = \{x \in \mathbb{R}^4 | x^1 > |x^0|\}.$$

It can then be proved that the local algebra $\mathcal{M}(\mathcal{W})$ corresponding to the wedge is a type III_1 factor (the (unique) hyperfinite type III_1 factor, if we assume an additional property, the split property, see e. g. [BW92, Chapter 16,17])). Furthermore, there is a uniquely defined one-parameter subgroup $\Lambda(t)$ of the Lorentz group (the *Lorentz boosts*) mapping $\mathcal{W}$ onto itself:

$$\Lambda(t) = \begin{pmatrix} \cosh(2\pi t) & -\sinh(2\pi t) & 0 & 0 \\ -\sinh(2\pi t) & \cosh(2\pi t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.4.2}$$

Under these conditions it can be shown that the vacuum Ω is also the unique invariant vector with respect to $\mathrm{U}(\Lambda(t))$ for all $t \in \mathbb{R}$ (For details see [BW92, § 7.3]).

The next theorem connects the modular objects for the wedge algebra with physical operators:

Theorem 2.4.2 (Bisognano and Wichmann). *Adopt the above notations. Assume that the local algebra $\mathcal{M}(\mathcal{W})$ for the wedge is generated by Wightman fields (see e. g. [Haa92] for the axioms of Wightman fields). Then the modular objects (Δ, J) corresponding to the vacuum Ω fulfil the following relations:*

$$\mathrm{J} = \Theta \mathrm{U}(\pi, e_1)$$
$$\Delta = \mathrm{U}(\Lambda(-i/2)),$$

where Θ is the PCT*-operator, $\mathrm{U}(\pi, e_1)$ represents the rotation around the x^1-axis by π, and $\Lambda(-i/2)$ is defined by* (2.4.2)*.*

For details and proofs we refer to [BW92, 16.1.1] or [Haa92, V.4.1], for instance.

Remark 2.4.3. Theorem 2.4.2 states that the modular operator Δ for the wedge algebra, which is a type III_1 factor, corresponding to the vacuum generates the representation of the Lorentz boosts. Furthermore, we know that the vacuum is the only vector invariant under this representation. Thus, Δ has only one eigenvector for the eigenvalue 1 (up to scalar multiples). Hence, we have an example for a type III_1 factor with modular operator possessing a one-dimensional eigenspace for the eigenvalue 1.

There are also other examples in which the modular objects act geometrically. For instance, Buchholz showed that the modular group of the forward light-cone coincides with the dilatations in a field theory of massless, noninteracting particles [Buc78]. Hislop and Longo computed the action of the modular group for the double cone in a conformal theory [HL82]. Borchers and Yngvason also investigated the action of the modular group for theories in thermal representations (see [BY99]).

Moreover, we say that a theory fulfils the *Bisognano-Wichmann property* if the modular group of every wedge acts as in Theorem 2.4.2. One can then prove further important results in the theory of local observables for theories fulfilling the Bisognano-Wichmann property, like e. g. the PCT theorem. For this and other applications see [Bor00] and the references therein.

Recent papers of Buchholz et. al. (see [BS93], [BDFS00], and [BMS00]) proposed some conditions involving the modular objects as selection criteria for significant states on arbitrary space-times, the so-called *Condition of Geometric Modular Action* and *Modular Stability Condition.*

For a further application, the inverse problems, we refer to Chapter 5.

Chapter 3

General Form of Modular Objects

In this chapter we will examine the modular objects in greater detail. Although there is a rich literature on modular theory, most authors restrict their attention only to modular automorphism group whereas we investigate the modular objects (modular operators and modular conjugations) themselves.

To this end, it is useful to introduce the concept of generalized vectors in §3.1. In the remaining sections we will investigate the modular objects separately for finite algebras, for infinite but still semifinite algebras, and for type III factors.

The contents of §3.1 can be found in the more general situation of algebras of unbounded operators in a series of papers by Inoue and others [IK94, Ino95b, Ino95a, Ino97, IKO99]. Lemma 3.1.8 is a generalization of a calculation which leads to the modular operator of derived weights to this more general context (see [PT73] or [Str81, § 4]).

The results of §3.2 and §3.3 seem to be more or less common knowledge. However, they can not be found explicitly in the literature in the form which is needed here. In particular, the correspondence between vectors in the Hilbert space on which a von Neumann algebra acts and operators affiliated with this von Neumann algebra (Lemma 3.2.3 and Lemma 3.3.5) can also be obtained by the use of the so-called Haagerup's correspondence [Haa79a] (see also [Yam92]). However, this construction is rather abstract such that the spatial meaning of this correspondence (Corollary 3.2.4 and Corollary 3.3.6) is not clear. In the finite case, however, Corollary 3.2.4 can be proved with the help of the so-called T theorem by Dye and Skau [Dye52, Ska77] (see also [Tak83, Theorem I.6.3]) which is not applicable to the infinite case. The form of the modular objects for finite von Neumann algebras (Theorem 3.2.7) can be found in the case of bounded invertible operators T_u in [KR86]. The unbounded case and the generalization to semifinite algebras (Theorem 3.3.9) are new. Note, however, that the form of the modular operator for a semifinite algebra can also be obtained by other techniques using the known form of the modular automorphism group (cf. Corollary 2.1.17) and seems to be common knowledge as well (see e. g. [Wol97]). However, the form of the modular conjugations for this more general case can not be found in the literature.

The contents of §3.4 are essentially new. However, the form of the modular operator corresponding to the dual of a trace (Proposition 3.4.3) can be found in [Sun87]. As in the semifinite case the form of the modular operator can also be obtained with the help of the known modular automorphism group (Theorem 2.2.8). Proceeding in this way one would not get the form of the modular conjugation. The type III_1 results seem to be new, especially Lemma 3.4.8 which was suggested by Wollenberg.

3.1 Generalized Vectors

In this section we present the concept of generalized vectors, a concept introduced by Inoue and Karwowski for algebras of unbounded operators (see [IK94], [Ino95b], [Ino95a], [Ino97], [IKO99]). Since it simplifies some proofs we also use it in the context of von Neumann algebras (of bounded operators).

Definition 3.1.1. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A linear map μ from a subset $\mathcal{D}(\mu)$ of $\mathcal{M}$ into $\mathcal{H}$ is said to be a *generalized vector* for $\mathcal{M}$ if the following conditions hold:

1. The domain $\mathcal{D}(\mu)$ of μ is a left ideal of $\mathcal{M}$.

2. $\mu(\mathrm{XA}) = \mathrm{X}\mu(\mathrm{A})$ for all $\mathrm{X} \in \mathcal{M}$ and $\mathrm{A} \in \mathcal{D}(\mu)$.

A generalized vector μ for $\mathcal{M}$ is *cyclic* if $\mu(\mathcal{D}(\mu))$ is dense in $\mathcal{H}$.

The following examples demonstrate that generalized vectors are a generalization of vectors and that they allow a description of weights analogous to the description of states as vector states:

Example 3.1.2. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$.

1. Let $v \in \mathcal{H}$. The map μ_v defined by $\mu_v(\mathrm{A}) := \mathrm{A}v$ for all $\mathrm{A} \in \mathcal{M}$ is a generalized vector for $\mathcal{M}$, and it is cyclic if and only if v is cyclic.

2. Assume that $\mathcal{H}$ is the GNS Hilbert space with respect to a n. s. f. weight φ. Define

$$\begin{aligned} \mu_\varphi : \mathcal{D}(\mu_\varphi) \subset \mathcal{M} &\to \mathcal{H} \\ \mathrm{A} &\mapsto \mu_\varphi(\mathrm{A}) := \mathrm{A}, \end{aligned}$$

   where $\mathcal{D}(\mu_\varphi) := \mathcal{N}_\varphi = \{\mathrm{A} \in \mathcal{M} | \varphi(\mathrm{A}^*\mathrm{A}) < \infty\}$ and $\mathcal{H} = \overline{\mathcal{N}_\varphi}^{\varphi}$. Then μ_φ is a cyclic generalized vector for $\mathcal{M}$.

Let now μ be a generalized vector for the von Neumann algebra $\mathcal{M}$ such that $\mu((\mathcal{D}(\mu)^*\mathcal{D}(\mu))^2)$ is dense in $\mathcal{H}$. Then μ is cyclic and we can define the *commutant of* μ (a generalized vector on the commutant of $\mathcal{M}$):

$$\begin{aligned} &\mathcal{D}(\mu^c) = \{\mathrm{A}^{'} \in \mathcal{M}^{'} | \exists v_{\mathrm{A}'} \in \mathcal{H} : \mathrm{A}^{'}\mu(\mathrm{X}) = \mathrm{X}v_{\mathrm{A}'} \quad \forall \mathrm{X} \in \mathcal{D}(\mu)\} \\ &\mu^c(\mathrm{A}^{'}) = v_{\mathrm{A}'} \quad \forall \mathrm{A}^{'} \in \mathcal{D}(\mu^c). \end{aligned}$$

Definition 3.1.3. Let $\mathcal{M}$, μ, μ^c be defined as above. Then μ is a *cyclic and separating generalized vector* for $\mathcal{M}$ if $\mu^c((\mathcal{D}(\mu^c)^*\mathcal{D}(\mu^c))^2)$ is dense in $\mathcal{H}$.

Example 3.1.4. We use the notation of Example 3.1.2.

1. Let v be a cyclic vector for $\mathcal{M}$. Then $\mathcal{D}(\mu_v^c) = \mathcal{M}^{'}$ and $\mu_v^c(\mathrm{A}^{'}) = \mathrm{A}v$ since
$$\mathrm{A}^{'}\mu_v(\mathrm{X}) = \mathrm{A}^{'}\mathrm{X}v = \mathrm{X}\mathrm{A}^{'}v$$
for all $\mathrm{X} \in \mathcal{D}(\mu_v) = \mathcal{M}$ and $\mathrm{A}^{'} \in \mathcal{M}^{'}$. Furthermore, μ_v is a cyclic and separating generalized vector if and only if v is cyclic and separating.

2. Note first that $\mu_\varphi((\mathcal{N}_\varphi^*\mathcal{N}_\varphi)^2)$ is dense in $\mathcal{H}$ since φ is semifinite (see [Str81, Theorem 2.2]). Let now $\mathfrak{A} := \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ and
$$\begin{aligned}\mathfrak{A}^{'} := \{z \in \mathcal{H} | \text{the mapping } m_z : \mathcal{H} \supset \mathfrak{A} &\to \mathcal{H} \\ \mathrm{A} &\mapsto m_z(\mathrm{A}) = \pi_\varphi(\mathrm{A})z \text{ is bounded}\},\end{aligned}$$
where
$$\begin{aligned}\pi_\varphi : \mathcal{M} &\to L(\mathcal{H}) \\ \mathrm{M} &\mapsto \pi_\varphi(\mathrm{M})\mathrm{B} = \mathrm{MB} \quad (\mathrm{B} \in \mathcal{N}_\varphi \subset \mathcal{H})\end{aligned}$$
is the GNS representation with respect to φ. Every element in $\mathfrak{A}^{'}$ then defines a bounded operator on $\mathcal{H}$ commuting with $\mathcal{M}$ and $\mathfrak{A}^{'} \subset \mathcal{D}(\mu_\varphi^c)$. Furthermore, $\mathrm{J}\mathfrak{A} = \mathfrak{A}^{'}$ where J is the modular conjugation with respect to φ (see [KR86, p. 654]). Since φ is semifinite, $\mathfrak{A}^2$ as well as
$$(\mathfrak{A}^{'})^2 = (\mathrm{J}\mathfrak{A})^2 = (\mathfrak{A})^2$$
are dense in $\mathcal{H}$. Thus, $(\mathcal{M}, \mu_\varphi)$ is a cyclic and separating generalized vector for $\mathcal{M}$.

Lemma 3.1.5 ([IK94]). *Let μ be a cyclic and separating generalized vector such that $\mu((\mathcal{D}(\mu)^* \cap \mathcal{D}(\mu))^2)$ and $\mu^c((\mathcal{D}(\mu^c)^* \cap \mathcal{D}(\mu^c))^2)$ are dense in $\mathcal{H}$. Let $\mu^{cc} \supset \mu$ denote the commutant of μ^c.*

1. *$\mu^c((\mathcal{D}(\mu^c)^* \cap \mathcal{D}(\mu^c))^2)$ is a right Hilbert algebra with right von Neumann algebra $\mathcal{M}^{'}$.*

2. *$\mu^{cc}((\mathcal{D}(\mu^{cc})^* \cap \mathcal{D}(\mu^{cc}))^2)$ is a left Hilbert algebra with left von Neumann algebra $\mathcal{M}^{'}$.*

3. *Let $\mathrm{S}_{\mu^{cc}}$ be the closure of the involution $\mu^{cc}(\mathrm{A}) \mapsto \mu^{cc}(\mathrm{A}^*)$ for $\mathrm{A} \in \mathcal{D}(\mu^{cc}) \cap \mathcal{D}(\mu^{cc})^*$ and let $\mathrm{S}_{\mu^{cc}} = \mathrm{J}_{\mu^{cc}}\Delta_{\mu^{cc}}^{1/2}$ be the polar decomposition of $\mathrm{S}_{\mu^{cc}}$. Then:*
   - *(a)* $\mathrm{J}_{\mu^{cc}}\mathcal{M}\mathrm{J}_{\mu^{cc}} = \mathcal{M}^{'}$.
   - *(b)* $\Delta_{\mu^{cc}}^{it}\mathrm{A}\Delta_{\mu^{cc}}^{-it} \in \mathcal{M}$ *for* $\mathrm{A} \in \mathcal{M}$.
   - *(c)* $\Delta_{\mu^{cc}}^{it}\mathrm{B}\Delta_{\mu^{cc}}^{-it} \in \mathcal{M}^{'}$ *for* $\mathrm{B} \in \mathcal{M}^{'}$.

Theorem 3.1.6 ([IK94]). *Let μ be as in Lemma 3.1.5, let S_μ denote the closure of the involution $\mu(\mathrm{A}) \mapsto \mu(\mathrm{A}^*)$ for $\mathrm{A} \in \mathcal{D}(\mu) \cap \mathcal{D}(\mu)^*$, and let $\mathrm{S}_\mu = \mathrm{J}_\mu \Delta_\mu^{1/2}$ be the polar decomposition of S_μ.*

1. $\mathrm{S}_\mu = \mathrm{S}_{\mu^{cc}}$.
2. $\sigma_t^\mu(\mathrm{A}) := \Delta_\mu^{it} \mathrm{A} \Delta_\mu^{-it}$ *($\mathrm{A} \in \mathcal{M}$, $t \in \mathbb{R}$) defines a one-parameter automorphism group of $\mathcal{M}$.*
3. *The weight φ_μ defined by*

$$\varphi_\mu(\mathrm{A}^*\mathrm{A}) := \langle \mu(\mathrm{A}) | \mu(\mathrm{A}) \rangle \qquad \text{for } \mathrm{A} \in \mathcal{D}(\mu)$$

satisfies the KMS-condition with respect to σ_t^μ.

Example 3.1.7. We use the notation of Example 3.1.2 and Example 3.1.4.

1. If v is cyclic and separating μ_v fulfils the conditions of Theorem 3.1.6 and $(\Delta_{\mu_v}, \mathrm{J}_{\mu_v})$ are the modular objects with respect to $(\mathcal{M}, v)$.
2. Example 3.1.4 implies that μ_φ fulfils the prerequisites of Theorem 3.1.6 as well. Then $(\Delta_{\mu_\varphi}, \mathrm{J}_{\mu_\varphi})$ are (unitarily equivalent to) the modular objects corresponding $(\mathcal{M}, \varphi)$.

Let now μ be a cyclic and separating generalized vector for a von Neumann algebra $\mathcal{M}$ with modular objects $(\Delta_\mu, \mathrm{J}_\mu)$ and let $\mathrm{A} \in \mathcal{D}(\mu) \cap \mathcal{M}^{\sigma^\mu}$ (i.e. $\mathrm{A} \in \mathcal{D}(\mu)$ commutes with Δ_μ^{it} for all $t \in \mathbb{R}$) be a bounded invertible operator. Setting $\mu_\mathrm{A} := \mu(\cdot \mathrm{A})$ we can define a vector $u_0 := \mu_\mathrm{A}(\mathrm{I}) = \mu(\mathrm{A})$ which is cyclic and separating for $\mathcal{M}$:

Lemma 3.1.8. *$u_0 := \mu_\mathrm{A}(\mathrm{I})$ is a cyclic and separating vector for $\mathcal{M}$ and the modular objects (Δ_0, J_0) corresponding to u_0 are*

$$\mathrm{J}_0 = \mathrm{J}_\mu \mathrm{V}^* \mathrm{J}_\mu \mathrm{V} \mathrm{J}_\mu = \mathrm{V} \mathrm{J}_\mu \mathrm{V}^*$$

and

$$\Delta_0 = \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0 \mathrm{H}_0 \Delta_\mu,$$

where $\mathrm{A} = \mathrm{H}_0^{1/2} \mathrm{V}$ is the polar decomposition of A.

Proof. 1. First note that the properties of cyclic and separating generalized vectors imply that $\mathcal{M}\mathrm{A} \subset \mathcal{D}(\mu)$ and

$$\mathrm{B}u_0 = \mathrm{B}\mu_\mathrm{A}(\mathrm{I}) = \mathrm{B}\mu(\mathrm{A}) = \mu(\mathrm{BA}).$$

Moreover, if $\mathrm{B} \in \mathcal{D}(\mu)$ then $\mathrm{B} = \mathrm{BA}^{-1}\mathrm{A} \in \mathcal{M}\mathrm{A}$, hence $\mathcal{D}(\mu) \subset \mathcal{M}\mathrm{A}$ and $\mathcal{D}(\mu) = \mathcal{M}\mathrm{A}$. The latter implies that $\mathcal{M}u_0 = \mu(\mathcal{M}\mathrm{A}) = \mu(\mathcal{D}(\mu))$ is dense in $\mathcal{H}$.

To compute the commutant of μ_A, let $\mathrm{A}' \in \mathcal{M}'$ and $\mathrm{X} \in \mathcal{M} = \mathcal{D}(\mu_\mathrm{A})$. Then

$$\mathrm{A}'\mu_\mathrm{A}(\mathrm{X}) = \mathrm{A}'\mu(\mathrm{XA}) = \mathrm{XA}'\mu(\mathrm{A}) = \mathrm{XA}'u_0,$$

hence $\mathrm{A}' \in \mathcal{D}(\mu^c_{\mathrm{A}})$ and $\mu^c_{\mathrm{A}}(\mathrm{A}') = \mathrm{A}'u_0$. If $\mathrm{A}' \in \mathcal{D}(\mu^c)$ and $\mathrm{X} \in \mathcal{M} = \mathcal{D}(\mu_{\mathrm{A}})$ we get

$$\mathrm{A}'\mu_{\mathrm{A}}(\mathrm{X}) = \mathrm{A}'\mu(\mathrm{XA}) = \mathrm{XA}\mu^c(\mathrm{A}'),$$

hence $\mathrm{A}' \in \mathcal{D}(\mu^c_{\mathrm{A}})$ and $\mu^c_{\mathrm{A}}(\mathrm{A}') = \mathrm{A}\mu^c(\mathrm{A}')$. This implies that

$$\mathcal{M}'u_0 = \mu^c_{\mathrm{A}}(\mathcal{D}(\mu^c_{\mathrm{A}})) \supset \mathrm{A}\mu^c(\mathcal{D}(\mu^c))$$

is also dense in $\mathcal{H}$. Therefore u_0 is a cyclic and separating vector for $\mathcal{M}$.

2. We show that the Tomita operator S_0 defined by

$$\mathrm{S}_0\mu_{\mathrm{A}}(\mathrm{B}) = \mu_{\mathrm{A}}(\mathrm{B}^*) \quad (\mathrm{B} \in \mathcal{M})$$

can be written as

$$\mathrm{S}_0 = \mathrm{H}^{-1}\mathrm{VSV}^*\mathrm{H}, \tag{3.1.1}$$

where $\mathrm{S} = \mathrm{J}_\mu\Delta_\mu^{1/2}$ is the Tomita operator corresponding to μ. To prove this let $\mathrm{B} \in \mathcal{M}$. Then

$$\begin{aligned}(\mathrm{H}^{-1}\mathrm{VSV}^*\mathrm{H})\mu_{\mathrm{A}}(\mathrm{B}) &= \mathrm{H}^{-1}\mathrm{VSV}^*\mathrm{H}\mu(\mathrm{BHV}) \\ &= \mathrm{H}^{-1}\mathrm{VS}\quad \underbrace{\mu(\mathrm{V}^*\mathrm{HBHV})}_{\in\mu(\mathcal{D}(\mu)\cap\mathcal{D}(\mu)^*)\subset\mathcal{D}(S)} \\ &= \mathrm{H}^{-1}\mathrm{V}\mu(\mathrm{V}^*\mathrm{HB}^*\mathrm{HV}) \\ &= \mu_{\mathrm{A}}(\mathrm{B}^*).\end{aligned}$$

Hence S_0 and $\mathrm{H}^{-1}\mathrm{VSV}^*\mathrm{H}$ coincide on $\mu_{\mathrm{A}}(\mathcal{M})$. Now $\mu_{\mathrm{A}}(\mathcal{M})$ is a core for S_0 by definition and, since $\mu(\mathcal{D}(\mu) \cap \mathcal{D}(\mu)^*)$ is a core for S,

$$\begin{aligned}\mu_{\mathrm{A}}(\mathcal{M}) = \mu(\mathcal{M}\mathrm{A}) = \mathrm{H}^{-1}\mathrm{V}\mu(\mathcal{M}\mathrm{A}) = \\ = \mathrm{H}^{-1}\mathrm{V}\mu(\mathcal{D}(\mu)) \supset \mathrm{H}^{-1}\mathrm{V}\mu(\mathcal{D}(\mu) \cap \mathcal{D}(\mu)^*)\end{aligned}$$

is also a core for $\mathrm{H}^{-1}\mathrm{VSV}^*\mathrm{H}$. This proves (3.1.1).

Moreover,

$$\begin{aligned}\mathrm{S}_0 &= \mathrm{H}^{-1}\mathrm{VJ}\Delta^{1/2}\mathrm{V}^*\mathrm{H} \\ &= \mathrm{H}^{-1}\mathrm{VJV}^*\mathrm{H}\Delta^{1/2} \\ &= \underbrace{\mathrm{VJV}^*}_{=\mathrm{J}_0}\underbrace{\mathrm{VJV}^*\mathrm{H}^{-1}\mathrm{VJV}^*\mathrm{H}\Delta^{1/2}}_{=\Delta_0^{1/2}}\end{aligned}$$

since $\mathrm{A} = \mathrm{HV}$ commutes with Δ and the polar decomposition is unique.

□

3.2 Finite Algebras

Throughout this section $\mathcal{M}$ is a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a cyclic, separating vector $u_0 \in \mathcal{H}$ and a (fixed) tracial state tr extended to all positive operators affiliated with $\mathcal{M}$ (cf. Remark 2.3.9). The aim of this section is to obtain a refinement of Corollary 2.1.17. To this end we first establish a correspondence between vectors in $\mathcal{H}$ and operators affiliated with $\mathcal{M}$ and prove some properties of this correspondence. Finally, we show how the modular objects corresponding to cyclic and separating vectors can be expressed with the help of these operators.

Since $\mathcal{M}$ has a cyclic and separating vector, it is in its standard form, and we can assume without loss of generality that $\mathcal{H}$ is the GNS space $L_2(\mathcal{M}, \mathrm{tr})$ of $(\mathcal{M}, \mathrm{tr})$ (cf. the remark following Theorem 2.1.12). In particular, this means that the scalar product $\langle \cdot|\cdot \rangle_2$ on $\mathcal{H} = L_2(\mathcal{M}, \mathrm{tr})$ is generated by the trace on $\mathcal{M}$:

$$\langle \mathrm{A}|\mathrm{B} \rangle_2 = \mathrm{tr}(\mathrm{B}^*\mathrm{A}) \quad \text{for } \mathrm{A}, \mathrm{B} \in \mathcal{M}$$

and the norm $\|\cdot\|_2$ coincides with the trace norm on $\mathcal{M}$. If we extend the trace to all positive closed operators affiliated with $\mathcal{M}$ we have

$$\mathrm{tr}(\mathrm{A}) = \lim_{n\to\infty} \mathrm{tr}(\mathrm{A}_n) = \sup_n \mathrm{tr}(\mathrm{A}_n)$$

for all sequences $(\mathrm{A}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ with $\mathrm{A}_n \uparrow \mathrm{A} \geq 0$ [Str81, Proposition 11.4] (see also Remark 2.3.9). In the following we establish a one-to-one correspondence between the set $\overline{\mathcal{N}_{\mathrm{tr}}} := \{\mathrm{T}\eta\mathcal{M} | \, \mathrm{tr}(\mathrm{T}^*\mathrm{T}) < \infty\}$ of square-integrable operators affiliated with $\mathcal{M}$ and $L_2(\mathcal{M}, \mathrm{tr})$.

We first prove the following property of operators in $\overline{\mathcal{N}_{\mathrm{tr}}}$:

Proposition 3.2.1. *Let* $\mathrm{T} \in \overline{\mathcal{N}_{\mathrm{tr}}}$. *Then* $\mathcal{M}$ *regarded as a subset of* $\mathcal{H} = L_2(\mathcal{M}, \mathrm{tr})$ *is a core for* T.

Proof. Let $\mathrm{T} \in \overline{\mathcal{N}_{\mathrm{tr}}}$, let $\mathrm{T} = \mathrm{VH}$ be its polar decomposition, and let $\mathrm{E}_M \in \mathcal{M}$ be the spectral measure of $\mathrm{H} \geq 0$. We set $\mathrm{H}_n := \mathrm{E}_{[0,n]}\mathrm{H}$. If $\mathrm{A} \in \mathcal{M} \subset L_2(\mathcal{M}, \mathrm{tr})$ we get

$$\|(\mathrm{VH}_n - \mathrm{VH}_m)\mathrm{A}\|_2^2 \leq \|\mathrm{A}\|^2 \, \mathrm{tr}((\mathrm{H}_n - \mathrm{H}_m)^2) \to 0$$

for $n, m \to \infty$. This implies $\mathrm{A} \in \mathcal{D}(\mathrm{T})$ and $\mathrm{TA} = \lim_{n\to\infty} \mathrm{VH}_n\mathrm{A}$ as vectors in $L_2(\mathcal{M}, \mathrm{tr})$.

Since H is positive, $(\mathrm{I} + \mathrm{H})^{-1} \in \mathcal{M}$ and the range of $(\mathrm{I} + \mathrm{H})^{-1}$ coincides with $\mathcal{D}(\mathrm{H})$. If $x \in \mathcal{D}(\mathrm{H}) = \mathcal{D}(\mathrm{T})$ there is a vector $y \in \mathcal{H} = L_2(\mathcal{M}, \mathrm{tr})$ such that $x = (\mathrm{I} + \mathrm{H})^{-1}y$. Since $\mathcal{M}$ is dense in $L_2(\mathcal{M}, \mathrm{tr})$ there is a sequence $(\mathrm{A}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ converging to y. Therefore, setting $\mathrm{B}_n := (\mathrm{I} + \mathrm{H})^{-1}\mathrm{A}_n \in \mathcal{M}$ we define a sequence $(\mathrm{B}_n)_{n\in\mathbb{N}}$ converging to x. Moreover, $\mathrm{TB}_n = \mathrm{VHB}_n$ converges to $\mathrm{V}(y - x)$ which implies that $\mathcal{M}$ is a core for T. □

In the next proposition we show that every converging sequence in $\mathcal{M} \subset \overline{\mathcal{M}} = L_2(\mathcal{M}, \mathrm{tr})$ defines an operator in $\overline{\mathcal{N}_{\mathrm{tr}}}$.

Proposition 3.2.2. *Let* $(\mathrm{T}_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}$ *which converges in* $L_2(\mathcal{M},\mathrm{tr})$ *with respect to the trace norm. The assignment*

$$\begin{aligned} \mathrm{T}_0 : \mathcal{M} \subset L_2(\mathcal{M},\mathrm{tr}) &\to L_2(\mathcal{M},\mathrm{tr}) \\ \mathrm{A} &\mapsto \mathrm{T}_0\mathrm{A} := \lim_{n\to\infty} \mathrm{T}_n\mathrm{A} \end{aligned} \tag{3.2.1}$$

defines a (linear) closable operator whose closure T *is affiliated with* $\mathcal{M}$. *Furthermore,* $\mathrm{tr}(\mathrm{T}^*\mathrm{T}) < \infty$.

Proof. 1. T_0 is well-defined since convergence of $(\mathrm{T}_n)_{n\in\mathbb{N}}$ in $L_2(\mathcal{M},\mathrm{tr})$ implies convergence of $(\mathrm{T}_n\mathrm{A})_{n\in\mathbb{N}}$ in $L_2(\mathcal{M},\mathrm{tr})$ for every (bounded) $\mathrm{A} \in \mathcal{M}$:

$$\begin{aligned} \|\mathrm{T}_n\mathrm{A} - \mathrm{T}_m\mathrm{A}\|_2^2 &= \mathrm{tr}(\mathrm{A}^*(\mathrm{T}_n^* - \mathrm{T}_m^*)(\mathrm{T}_n - \mathrm{T}_m)\mathrm{A}) \\ &\le \|\mathrm{A}\|^2\,\mathrm{tr}((\mathrm{T}_n^* - \mathrm{T}_m^*)(\mathrm{T}_n - \mathrm{T}_m)) \to 0 \quad \text{for } n,m\to\infty. \end{aligned} \tag{3.2.2}$$

2. Let T_0^* be defined as T_0 with $(\mathrm{T}_n^*)_{n\in\mathbb{N}}$ instead of $(\mathrm{T}_n)_{n\in\mathbb{N}}$ (note that $\mathrm{tr}(\mathrm{A}^*\mathrm{A}) = \mathrm{tr}(\mathrm{A}\mathrm{A}^*)$ for all $\mathrm{A} \in \mathcal{M}$, such that $(\mathrm{T}_n^*)_{n\in\mathbb{N}}$ also converges in $L_2(\mathcal{M},\mathrm{tr})$). Then

$$\langle \mathrm{T}_0\mathrm{A}|\mathrm{B}\rangle_2 = \lim_{n\to\infty}\langle \mathrm{T}_n\mathrm{A}|\mathrm{B}\rangle_2 = \lim_{n\to\infty}\langle \mathrm{A}|\mathrm{T}_n^*\mathrm{B}\rangle_2 = \langle \mathrm{A}|\mathrm{T}_0^*\mathrm{B}\rangle_2.$$

for $\mathrm{A},\mathrm{B} \in \mathcal{M}$. It follows that $(\mathrm{T}_0)^* \supset \mathrm{T}_0^*$ and, since T_0^* is densely defined, T_0 is closable.

3. The closure T of T_0 is affiliated with $\mathcal{M}$. To prove this, let U' be a unitary in $\mathcal{M}'$. Since $\mathcal{M}$ is in standard form and finite there is a unitary $\mathrm{V} \in \mathcal{M}$ such that $\mathrm{U}'\mathrm{A} = \mathrm{A}\mathrm{V}$ for all $\mathrm{A} \in \mathcal{M}$ (see [KR86, Theorem 7.2.15]). Let now $(\mathrm{A}_k)_{k\in\mathbb{N}}$ denote a sequence in $\mathcal{M}$ such that

$$\lim_{k\to\infty}\mathrm{A}_k = x \in \mathcal{D}(\mathrm{T}) \subset L_2(\mathcal{M},\mathrm{tr}) \quad \text{and} \quad \lim_{k\to\infty}\mathrm{T}\mathrm{A}_k = \mathrm{T}x$$

both with respect to the trace norm $\|\cdot\|_2$. Then

$$\left\|\mathrm{T}\mathrm{U}'\mathrm{A}_k - \mathrm{T}\mathrm{U}'\mathrm{A}_l\right\|_2 = \|\mathrm{T}\mathrm{A}_k\mathrm{V} - \mathrm{T}\mathrm{A}_l\mathrm{V}\|_2 = \|\mathrm{T}\mathrm{A}_k - \mathrm{T}\mathrm{A}_l\|_2\,,$$

and thus $\mathrm{U}'x \in \mathcal{D}(\mathrm{T})$ with $\lim_{k\to\infty}\mathrm{T}\mathrm{U}'\mathrm{A}_k = \mathrm{T}\mathrm{U}'x$. Moreover,

$$\begin{aligned} \mathrm{T}\mathrm{U}'x &= \lim_{k\to\infty}\lim_{n\to\infty}\mathrm{T}_n\mathrm{U}'\mathrm{A}_k \\ &= \lim_{k\to\infty}\lim_{n\to\infty}\mathrm{T}_n\mathrm{A}_k\mathrm{V} \\ &= \lim_{k\to\infty}\lim_{n\to\infty}\mathrm{U}'\mathrm{T}_n\mathrm{A}_k \\ &= \mathrm{U}'\mathrm{T}x. \end{aligned}$$

Similarly, $\mathrm{U}'x \in \mathcal{D}(\mathrm{T})$ implies $x \in \mathcal{D}(\mathrm{T})$. Hence, $\mathcal{D}(\mathrm{T}\mathrm{U}') = \mathcal{D}(\mathrm{T})$ and T is affiliated with $\mathcal{M}$.

4. Furthermore, we get

$$\mathrm{tr}(\mathrm{T}^*\mathrm{T}) = \lim_{n\to\infty} \|\mathrm{T}_n\|_2^2 < \infty,$$

hence $\mathrm{T} \in \overline{\mathcal{N}_{\mathrm{tr}}}$. □

The same calculation as in (3.2.2) now implies that a sequence converging to 0 in $L_2(\mathcal{M}, \mathrm{tr})$ corresponds to $0 \in \mathcal{M}$. Therefore, the correspondence established by Proposition 3.2.2 is between vectors in $L_2(\mathcal{M}, \mathrm{tr})$ and $\overline{\mathcal{N}_{\mathrm{tr}}}$. It is injective. Indeed, let $(\mathrm{T}_n)_{n\in\mathbb{N}}$ and $(\mathrm{S}_n)_{n\in\mathbb{N}}$ denote two sequences in $\mathcal{M}$ converging in $L_2(\mathcal{M}, \mathrm{tr})$ such that the corresponding operators T and $\mathrm{S} \in \overline{\mathcal{N}_{\mathrm{tr}}}$ are equal. We obtain

$$\lim_{n\to\infty} \mathrm{T}_n\mathrm{I} = \mathrm{TI} = \mathrm{SI} = \lim_{n\to\infty} \mathrm{S}_n\mathrm{I} \tag{3.2.3}$$

since the identity $\mathrm{I} \in \mathcal{M}$ is in $\mathcal{D}(\mathrm{T}) \cap \mathcal{D}(\mathrm{S})$ (cf. Proposition 3.2.1). Equality (3.2.3) implies $\lim_{n\to\infty} \mathrm{tr}((\mathrm{T}_n - \mathrm{S}_n)^*(\mathrm{T}_n - \mathrm{S}_n)) = 0$, i. e. $\lim_{n\to\infty} \mathrm{T}_n = \lim_{n\to\infty} \mathrm{S}_n$ in $L_2(\mathcal{M}, \mathrm{tr})$.

On the other hand, let $\mathrm{T} \in \overline{\mathcal{N}_{\mathrm{tr}}}$, let $\mathrm{T} = \mathrm{VH}$ be its polar decomposition, and let $\mathrm{E}_M \in \mathcal{M}$ be the spectral measure of $\mathrm{H} \geq 0$. Setting $\mathrm{H}_n := \mathrm{E}_{[0,n]}\mathrm{H}$ we get a sequence $(\mathrm{VH}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ converging in $L_2(\mathcal{M}, \mathrm{tr})$ since

$$\mathrm{tr}((\mathrm{H}_n\mathrm{V}^* - \mathrm{H}_m\mathrm{V}^*)(\mathrm{VH}_n - \mathrm{VH}_n)) = \mathrm{tr}((\mathrm{H}_n - \mathrm{H}_m)^2) = \mathrm{tr}(\mathrm{H}_n^2 - \mathrm{H}_m^2) \to 0$$

for $n, m \to \infty$.

Furthermore, let $\tilde{\mathrm{T}}$ be the operator defined by (3.2.1) with the sequence $(\mathrm{VH}_n)_n$ instead of $(\mathrm{T}_n)_n$. According to Proposition 3.2.1, we have $\mathrm{TA} = \lim_{n\to\infty} \mathrm{VH}_n\mathrm{A} = \tilde{\mathrm{T}}\mathrm{A}$ for $\mathrm{A} \in \mathcal{M} \subset \mathcal{D}(\mathrm{T})$. Since $\mathcal{M}$ is a core for T the two operators $\mathrm{T}, \tilde{\mathrm{T}}$ coincide. We have thus constructed a bijective correspondence between vectors in $L_2(\mathcal{M}, \mathrm{tr})$ and $\overline{\mathcal{N}_{\mathrm{tr}}}$.

In the following lemma, we summarize the results obtained so far.

Lemma 3.2.3. *There is a bijective correspondence between vectors in $L_2(\mathcal{M}, \mathrm{tr})$ and operators in $\overline{\mathcal{N}_{\mathrm{tr}}}$. We denote the operator corresponding to a vector u by T_u.*

In the following we will always write $L_2(\mathcal{M}, \mathrm{tr})$ for $\overline{\mathcal{N}_{\mathrm{tr}}}$. The action of $\mathcal{M}$ (and $L_2(\mathcal{M}, \mathrm{tr})$) on $L_2(\mathcal{M}, \mathrm{tr})$ is multiplication from the left. Since in the GNS representation the trace is generated by a cyclic vector u_{tr} which corresponds to the identity I in $\mathcal{M}$ we have now $\mathrm{T}_u u_{\mathrm{tr}} = u$ for every $u \in \mathcal{H}$. Furthermore, T_u is the only operator with this property since u_{tr} is separating. We have thus proved:

Corollary 3.2.4. *For every vector $u \in \mathcal{H}$ there exists exactly one operator $\mathrm{T}_u \in L_2(\mathcal{M}, \mathrm{tr})$ affiliated with $\mathcal{M}$ such that $\mathrm{T}_u u_{\mathrm{tr}} = u$.*

Remark 3.2.5. 1. The operator T_u can also be obtained by the so called "T theorem" by Dye and Skau (see e. g. [Tak83, Theorem 6.3], cf. also [Bol00a]). We chose our approach because it is also applicable in the infinite, semifinite case.

2. Let $\omega := \langle \cdot u | u \rangle$ be the normal state generated by a cyclic and separating vector u. Set $\mathrm{B} := \mathrm{T}_u \mathrm{T}_u^* \eta \mathcal{M}$. Then
$$\begin{aligned}\omega(\mathrm{A}) &= \langle \mathrm{A}u|u\rangle = \langle \mathrm{A}\mathrm{T}_u u_{\mathrm{tr}} | \mathrm{T}_u u_{\mathrm{tr}} \rangle = \mathrm{tr}(\mathrm{T}_u^* \mathrm{A} \mathrm{T}_u) \\ &= \mathrm{tr}(\mathrm{T}_u \mathrm{T}_u^* \mathrm{A}) = \mathrm{tr}_{\mathrm{B}}(\mathrm{A}) \qquad \text{for all } \mathrm{A} \in \mathcal{M},\end{aligned}$$
and Theorem 2.1.16 implies that $[D\omega : D\,\mathrm{tr}]_t = \mathrm{B}^{it}$.

Lemma 3.2.6. *Let u be a vector in $\mathcal{H}$. Suppose that $\mathrm{T}_u \in L_2(\mathcal{M}, \mathrm{tr})$ is the operator corresponding to u such that $u = \mathrm{T}_u u_{\mathrm{tr}}$. Then the following assertions are equivalent:*

(i) T_u *is injective.*

(ii) u *is cyclic.*

(iii) u *is separating.*

(iv) T_u *has dense range.*

Proof. Let $\mathrm{T}_u = \mathrm{VH}$ be the polar decomposition of T_u where $\mathrm{V} \in \mathcal{M}$ is a partial isometry and $\mathrm{H}\eta\mathcal{M}$ is positive.

$(i) \Leftrightarrow (ii)$**:** Assume that T_u is injective, and set
$$\mathrm{T}_n := \mathrm{E}_{[1/n,n]} \mathrm{H}^{-1} \mathrm{V}^* \in \mathcal{M},$$
where E_M is the spectral measure of H. If $\mathrm{M} \in \mathcal{M}$ then $\mathrm{MT}_n\mathrm{T}_u$ tends to M in $L_2(\mathcal{M}, tr)$, hence $\mathcal{M}\mathrm{T}_u$ is dense in $L_2(\mathcal{M}, tr)$ since $\mathcal{M}$ is dense, and u is cyclic.

If T_u is not injective, let $0 \neq \mathrm{E} \in \mathcal{M}$ be the projection onto the kernel of T_u. If u was cyclic there would be a sequence M_n such that $\mathrm{M}_n\mathrm{T}_u$ would tend to I in $L_2(\mathcal{M}, \mathrm{tr})$, but then $\mathrm{M}_n\mathrm{T}_u\mathrm{E} = 0$ would also tend to $\mathrm{E} \neq 0$ which is a contradiction.

$(iii) \Leftrightarrow (iv)$**:** u is separating, if and only if $\mathrm{M}u = 0$ for $\mathrm{M} \in \mathcal{M}$ implies $\mathrm{M} = 0$. The latter holds if and only if $\mathrm{MT}_u = 0$ for $\mathrm{M} \in \mathcal{M}$ implies $\mathrm{M} = 0$ which holds if and only if T_u has dense range.

$(iv) \Leftrightarrow (i)$**:** T_u has dense range if and only if $\mathrm{VV}^* = \mathrm{I}$. Since $\mathcal{M}$ is finite the latter holds if and only if $\mathrm{V}^*\mathrm{V} = \mathrm{I}$, i.e. if and only if H has dense range. Since H is selfadjoint it has dense range if and only if it is injective, which holds if and only if T_u is injective. □

Now we investigate how the modular objects of the cyclic and separating vector $u_0 \in \mathcal{H}$ for a finite von Neumann algebra $\mathcal{M}$ are related to the operator T_{u_0}.

Let again u_{tr} be a fixed cyclic trace vector for the finite algebra $\mathcal{M}$. We can then define a conjugation J (the modular conjugation corresponding to u_{tr}) by
$$\begin{aligned}&\mathrm{J} : \mathcal{H} \to \mathcal{H} \\ &\mathrm{A}u_{\mathrm{tr}} \mapsto \mathrm{JA}u_{\mathrm{tr}} = \mathrm{A}^* u_{\mathrm{tr}}.\end{aligned} \tag{3.2.4}$$
Note that $\mathrm{A} \mapsto \mathrm{JA}^*\mathrm{J}$ is an anti-isomorphism from $\mathcal{M}$ onto $\mathcal{M}'$.

Theorem 3.2.7. *Let $\mathcal{M}$ be a finite von Neumann algebra with cyclic and separating vector $u_0 \in \mathcal{M}$ and cyclic trace vector $u_{\mathrm{tr}} \in \mathcal{H}$. Let further $\mathrm{T}_{u_0} \eta \mathcal{M}$ be the invertible operator corresponding to u_0 with $\mathrm{tr}(\mathrm{T}_u^* \mathrm{T}_u) < \infty$. Let $\mathrm{T}_{u_0} = \mathrm{H}_0^{1/2}\mathrm{V}$ be the polar decomposition of T_{u_0}. Then the modular objects (Δ_0, J_0) of $(\mathcal{M}, u_0)$ are*

$$\mathrm{J}_0 = \mathrm{J}\mathrm{V}^*\mathrm{J}\mathrm{V}\mathrm{J} = \mathrm{V}\mathrm{J}\mathrm{V}^*,$$

where J *is the conjugation defined by* (3.2.4)*, and*

$$\Delta_0 = \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0 \mathrm{H}_0,$$

where the product is the closure of the product of the two commuting operators H_0 *and* $\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$.

Proof. Let $\mathrm{E}_n := \mathrm{E}_{[1/n,n]} \in \mathcal{M}$ be the spectral projections of H corresponding to the interval $[1/n, n]$. Then $\mathrm{T}_n := (\mathrm{H}\mathrm{E}_n + (\mathrm{I} - \mathrm{E}_n))\mathrm{V}$ is in $\mathcal{M}$. Defining $u_n := \mathrm{T}_n u_{\mathrm{tr}}$ we obtainb a sequence of cyclic and separating vectors (T_n are invertible operators in $\mathcal{M}$, cf. Lemma 3.2.3) converging to u_0 with modular objects

$$\begin{aligned} \mathrm{J}_n &= \mathrm{J}\mathrm{V}^*\mathrm{J}\mathrm{V}\mathrm{J} = \mathrm{V}\mathrm{J}\mathrm{V}^* \text{ and} \\ \Delta_n &= \mathrm{J}\mathrm{V}^*(\mathrm{H}\mathrm{E}_n + (\mathrm{I} - \mathrm{E}_n))^{-2}\mathrm{V}\mathrm{J}(\mathrm{H}\mathrm{E}_n + (\mathrm{I} - \mathrm{E}_n))^2, \end{aligned}$$

where J is the conjugation corresponding to the trace vector u_{tr} defined by (3.2.4) (see [KR86, 9.6.11]).

By Theorem 2.1.8 the modular conjugation of $u_0 = \lim_{n\to\infty} u_n$ is $\mathrm{J}\mathrm{V}^*\mathrm{J}\mathrm{V}\mathrm{J} = \mathrm{V}\mathrm{J}\mathrm{V}^*$ since all u_n lie in the same natural (closed) cone. Furthermore, the modular groups Δ_n^{it} corresponding to u_n converge in the strong operator topology to the modular group Δ_0^{it} corresponding to u_0 (cf. [Str81, p.106]). Since

$$\Delta_n^{it} = \mathrm{J}\mathrm{V}^*(\mathrm{H}\mathrm{E}_n + (\mathrm{I} - \mathrm{E}_n))^{2it}\mathrm{V}\mathrm{J}(\mathrm{H}\mathrm{E}_n + (\mathrm{I} - \mathrm{E}_n))^{2it}$$

and operator multiplication is continuous on bounded sets with respect to the strong operator topology, we have

$$\begin{aligned} \Delta_0^{it} &= so - \lim_{n\to\infty} \Delta_n^{it} \\ &= \mathrm{J}\mathrm{V}^*\mathrm{H}^{2it}\mathrm{V}\mathrm{J}\mathrm{H}^{2it} \\ &= \mathrm{V}\mathrm{J}\mathrm{V}^*\mathrm{H}^{2it}\mathrm{V}\mathrm{J}\mathrm{V}^*\mathrm{H}^{2it} \\ &= \mathrm{J}_0\mathrm{H}_0^{it}\mathrm{J}_0\mathrm{H}_0^{it}. \end{aligned}$$

Since $\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$ and H_0 commute, $\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0 \cdot \mathrm{H}_0$ is closable (cf. [KR83, 5.6.15]) and the closure $\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\mathrm{H}_0$ is selfadjoint such that $\Delta_0 = \mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\mathrm{H}_0$. □

3.3 Properly Infinite, Semifinite Algebras

Throughout this section $\mathcal{M}$ is a semifinite, properly infinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a cyclic, separating vector $u_0 \in \mathcal{H}$ and a (fixed) tracial weight tr extended to all positive operators affiliated with

$\mathcal{M}$. The aim of this section is to obtain a refinement of Corollary 2.1.17 in this case as well. To this end, we first establish a correspondence between vectors in $\mathcal{H}$ and operators affiliated with $\mathcal{M}$ and prove some properties of this correspondence in analogy to §3.2. Finally, we show how the modular objects corresponding to cyclic and separating vectors can be expressed with the help of these operators.

We first cite the following result [KR86, Proposition 6.3.12].

Proposition 3.3.1. *Let $\mathcal{M}$ be a type II_∞ von Neumann algebra. Then there is a countable family of finite equivalent projections with sum* I.

In the type I_∞ case the family of countably many abelian projections with central carriers I and sum I is a countable family of finite equivalent projections. Moreover, a semifinite, properly infinite von Neumann algebra is by definition the direct sum of a type I_∞ and a type II_∞ algebra. Hence, the following theorem can be proved for all semifinite, properly infinite von Neumann algebras:

Theorem 3.3.2. *Let $\mathcal{M}$ be a properly infinite, semifinite von Neumann algebra. Then there is a finite algebra $\mathcal{N}$ such that $\mathcal{M}$ is isomorphic to $\mathcal{N} \otimes L(\mathcal{K})$, where $\mathcal{K}$ is an infinite dimensional Hilbert space.*

Corollary 3.3.3. *If $\mathcal{M}$ is of type I_∞ the finite algebra $\mathcal{N}$ in Theorem 3.3.2 can be chosen to be of type I_1. If $\mathcal{M}$ is of type II_∞ the finite algebra in Theorem 3.3.2 is of type II_1.*

Proofs can be found in [KR86, Theorem 6.7.10].

As in §3.2 we again assume without loss of generality that $\mathcal{H}$ is the GNS space $L_2(\mathcal{M}, \mathrm{tr})$ with respect to $(\mathcal{M}, \mathrm{tr})$ and that the scalar product $\langle\cdot|\cdot\rangle_2$ on $\mathcal{H} = L_2(\mathcal{M}, \mathrm{tr})$ is generated by the trace on $\mathcal{N}_{\mathrm{tr}} \subset \mathcal{M}$. We can now prove the following results in the same way as the corresponding results of §3.2 substituting $\mathcal{N}_{\mathrm{tr}}$ for $\mathcal{M}$.

Proposition 3.3.4. *Let $(\mathrm{T}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{N}_{\mathrm{tr}}$ which converges in $L_2(\mathcal{M}, \mathrm{tr})$ with respect to the trace norm. The assignment*

$$\begin{aligned} \mathrm{T}_0 : \mathcal{N}_{\mathrm{tr}} \subset L_2(\mathcal{M}, \mathrm{tr}) &\to L_2(\mathcal{M}, \mathrm{tr}) \\ \mathrm{A} &\mapsto \mathrm{T}_0\mathrm{A} := \lim_{n\to\infty} \mathrm{T}_n\mathrm{A} \end{aligned} \tag{3.3.1}$$

defines a (linear) closable operator whose closure T *is affiliated with $\mathcal{M}$. Furthermore,* $\mathrm{tr}(\mathrm{T}^*\mathrm{T}) < \infty$.

Lemma 3.3.5. *There is a bijective correspondence between vectors in $L_2(\mathcal{M}, \mathrm{tr})$ and operators in $\overline{\mathcal{N}_{\mathrm{tr}}}$. We denote the operator corresponding to a vector u by T_u.*

In the following we will always write $L_2(\mathcal{M}, \mathrm{tr})$ for $\overline{\mathcal{N}_{\mathrm{tr}}}$. The action of $\mathcal{N}_{\mathrm{tr}}$ (and $L_2(\mathcal{M}, \mathrm{tr})$) on $L_2(\mathcal{M}, \mathrm{tr})$ is multiplication from the left.

Let now $(\mathrm{E}_n)_{n\in\mathbb{N}}$ be a countable orthogonal family of pairwise equivalent, finite projections with sum I. Such a family exists by Proposition 3.3.1. Then $\mathrm{tr} = \sum_n \mathrm{tr}_n$, where $\mathrm{tr}_n := \mathrm{tr}(\mathrm{E}_n \cdot \mathrm{E}_n)$ are traces on the finite algebras

$\mathcal{M}_n := \mathrm{E}_n \mathcal{M} \mathrm{E}_n$. In the GNS representation the trace is generated by the sequence of vectors v_n which correspond to the projections E_n in $\mathcal{M}$, because

$$\sum_n \langle \mathrm{A} v_n | v_n \rangle = \sum_n \operatorname{tr}(\mathrm{E}_n \mathrm{A} \mathrm{E}_n) = \sum_n \operatorname{tr}_n(\mathrm{A}) = \operatorname{tr}(\mathrm{A}) \quad (\mathrm{A} \in \mathcal{F}_{tr}).$$

Furthermore, since $\sum \mathrm{E}_n = \mathrm{I}$, we have $\sum_n \mathrm{T}_u v_n = u$ for every $u \in \mathcal{H}$ and T_u is the only operator with this property by Lemma 3.3.5. We have thus proved:

Corollary 3.3.6. *For every vector $u \in \mathcal{H}$ there exists exactly one operator $\mathrm{T}_u \in L_2(\mathcal{M}, \operatorname{tr})$ affiliated with $\mathcal{M}$ such that $\sum_n \mathrm{T}_u v_n = u$.*

Remark 3.3.7.

1. As in Remark 3.2.5.2 $\mathrm{B} := \mathrm{T}_u \mathrm{T}_u^* \eta \mathcal{M}$ is the positive operator such that $[D\omega : D \operatorname{tr}]_t = \mathrm{B}^{it}$ where $\omega := \langle \cdot u | u \rangle$ is the normal state generated by a cyclic and separating vector u.
2. Another way to get the operator T_u and the sequence $(v_n)_{n \in \mathbb{N}}$ defined above is to use the tensor product decomposition of II_∞ algebras in a finite algebra and a I_∞ factor (see Theorem 3.3.2) and the results of §3.2 for finite algebras. This way was considered in [Bol00b].

Lemma 3.3.8. *Let u be a vector in $\mathcal{H}$. Suppose that $\mathrm{T}_u \in L_2(\mathcal{M}, \operatorname{tr})$ is the operator corresponding to u such that $u = \sum_n \mathrm{T}_u v_n$. Then the following assertions hold:*

(i) T_u *is injective if and only if u is cyclic.*

(ii) T_u *has dense range if and only if u is separating.*

Proof. The proofs are exactly the same as those of the corresponding assertions in Lemma 3.2.6. □

Now we investigate how the modular objects of the cyclic and separating vector $u_0 \in \mathcal{H}$ for a semifinite, properly infinite von Neumann algebra $\mathcal{M}$ are related to the operator T_{u_0}.

The family $(v_n)_{n \in \mathbb{N}}$ defined above is an orthogonal cyclic family both for $\mathcal{M}$ and $\mathcal{M}'$ (an *orthogonal cyclic family* for a von Neumann algebra $\mathcal{M}$ is a family of vectors such that $[\mathcal{M} v_n]$ are mutually orthogonal and $\sum_n [\mathcal{M} v_n] = \mathrm{I}$, see [RVDV77]). Indeed,

$$\langle \mathrm{M} v_n | \mathrm{N} v_m \rangle = \operatorname{tr}(\mathrm{E}_m \mathrm{N}^* \mathrm{M} \mathrm{E}_n) = \delta_{n,m} \operatorname{tr}_n(\mathrm{N}^* \mathrm{M})$$

for $m, n \in \mathbb{N}$ and $\mathrm{M}, \mathrm{N} \in \mathcal{M}$. Hence $[\mathcal{M} v_m] \perp [\mathcal{M} v_m]$ for $n \neq m$, and $\sum_n [\mathcal{M} v_n] = \mathrm{I}$ since $\sum_n \mathrm{E}_n = \mathrm{I}$. On the other hand, since $\mathcal{M}$ is in standard form and semifinite, we have

$$[\mathcal{M}' v_n] = [\mathcal{M}' \mathrm{E}_n] = [\mathrm{E}_n \mathcal{M}] = \mathrm{E}_n.$$

Hence, $(v_n)_n$ is an orthogonal cyclic family for $\mathcal{M}'$ as well.

Defining

$$\begin{aligned} \mu_0 : \mathcal{D}(\mu_0) \subset \mathcal{M} &\to \mathcal{H} \\ \mathrm{M} &\mapsto \mu_0(\mathrm{M}) := \sum_n \mathrm{M} v_n, \end{aligned} \tag{3.3.2a}$$

with domain

$$\mathcal{D}(\mu_0) = \mathcal{N}_{\mathrm{tr}} = \{\mathrm{M} \in \mathcal{M} |\, \mathrm{tr}(\mathrm{M}^*\mathrm{M}) = \sum_n \|\mathrm{M}v_n\|^2 < \infty\} \tag{3.3.2b}$$

we obtain a cyclic generalized vector for $\mathcal{M}$ such that

$$\mu_0((\mathcal{D}(\mu_0)^* \cap \mathcal{D}(\mu_0))^2) = \mu_0(\mathcal{D}(\mu_0)^2)$$

is dense in $\mathcal{H}$ (cf. Example 3.1.4). Let further

$$\begin{aligned}\mathcal{D}(\mu_0^{\mathrm{c}}) &= \{\mathrm{M}' \in \mathcal{M}' | \exists v_{\mathrm{M}'} : \mathrm{M}'\mu_0(\mathrm{X}) = \mathrm{X}v_{\mathrm{M}'} \quad \forall \mathrm{X} \in \mathcal{D}(\mu_0)\} \\ \mu_0^{\mathrm{c}}(\mathrm{M}') &= v_{\mathrm{M}'} \quad \forall \mathrm{M}' \in \mathcal{D}(\mu_0^{\mathrm{c}})\end{aligned}$$

be the commutant of μ_0. Fix an $\mathrm{M}' \in \mathcal{M}'$ such that $\sum_n \left\|\mathrm{M}'v_n\right\|^2 < \infty$. Then

$$\mathrm{M}'\sum_n \mathrm{X}v_n = \sum_n \mathrm{X}\mathrm{M}'v_n = \mathrm{X}\sum_n \mathrm{M}'v_n$$

for all $\mathrm{X} \in \mathcal{D}(\mu_0)$, hence $\mathrm{M}' \in \mathcal{D}(\mu_0^{\mathrm{c}})$ and $\mu_0^{\mathrm{c}}(\mathrm{M}') = \sum_n \mathrm{M}'v_n$.

On the other hand, if $\mathrm{M}' \in \mathcal{D}(\mu_0^{\mathrm{c}})$ then

$$\begin{aligned}\sum_n \left\|\mathrm{M}'v_n\right\|^2 &= \sum_n \left\|\mathrm{M}'\sum_j \mathrm{E}_j v_n\right\|^2 = \sum_n \left\|\mathrm{M}'\sum_j \mathrm{E}_n v_j\right\|^2 \\ &= \sum_n \left\|\mathrm{M}'\mu_0(\mathrm{E}_n)\right\|^2 = \sum_n \|\mathrm{E}_n v_{\mathrm{M}'}\|^2 = \|v_{\mathrm{M}'}\|^2 < \infty\end{aligned}$$

since $\mathrm{E}_n = [\mathcal{M}'v_n] \in \mathcal{N}_{\mathrm{tr}} = \mathcal{D}(\mu_0)$ and $\sum_n \mathrm{E}_n = \mathrm{I}$. Thus, we have shown

$$\begin{aligned}\mathcal{D}(\mu_0^{\mathrm{c}}) &= \{\mathrm{M}' \in \mathcal{M}' : \sum_n \left\|\mathrm{M}'v_n\right\|^2 < \infty\} \\ \mu_0^{\mathrm{c}}(\mathrm{M}') &= \sum_n \mathrm{M}'v_n.\end{aligned}$$

This implies that μ_0^{c} is also a cyclic generalized vector for $\mathcal{M}'$ such that

$$\mu_0^{\mathrm{c}}((\mathcal{D}(\mu_0^{\mathrm{c}})^* \cap \mathcal{D}(\mu_0^{\mathrm{c}}))^2) = \mu_0^{\mathrm{c}}(\mathcal{D}(\mu_0^{\mathrm{c}})^2)$$

is dense in $\mathcal{H}$ (cf. Example 3.1.4). Thus μ_0 is a cyclic and separating generalized vector for $\mathcal{M}$ fulfilling the prerequisites of Lemma 3.1.5 and Theorem 3.1.6. Since $\mathrm{tr}(\mathrm{A}^*\mathrm{A}) = \langle \mu_0(\mathrm{A}) | \mu_0(\mathrm{A}) \rangle$ for $\mathrm{A} \in \mathcal{N}_{tr}$ the modular objects (Δ, J) are (unitarily equivalent to) those of tr and therefore $\Delta = \mathrm{I}$. Using the modular conjugation J corresponding to the cyclic and separating generalized vector μ_0 we define an anti-isomorphism from $\mathcal{M}$ onto $\mathcal{M}'$ by $\mathrm{A} \mapsto \mathrm{J}\mathrm{A}^*\mathrm{J}$.

Theorem 3.3.9. *Let $\mathcal{M}$ be a semifinite von Neumann algebra with cyclic and separating vector $u_0 \in \mathcal{M}$. Let further $\mathrm{T}_{u_0}\eta\mathcal{M}$ be the invertible operator corresponding to u_0 with $\mathrm{tr}(\mathrm{T}_u\mathrm{T}_u^*) < \infty$. Let $\mathrm{T}_{u_0} = \mathrm{H}_0^{1/2}\mathrm{V}$ be the polar decomposition of T_{u_0}. Then the modular objects (Δ_0, J_0) of $(\mathcal{M}, u_0)$ are*

$$\mathrm{J}_0 = \mathrm{J}\mathrm{V}^*\mathrm{J}\mathrm{V}\mathrm{J} = \mathrm{V}\mathrm{J}\mathrm{V}^*,$$

where J *is the above defined conjugation of* μ_0, *and*

$$\Delta_0 = \mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\mathrm{H}_0,$$

where the product is the closure of the product of the two commuting operators H_0 *and* $\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$.

Proof. 1. The finite case was treated in Theorem 3.2.7.

2. Assume first that $\mathrm{T}_{u_0}, \mathrm{T}_{u_0}^{-1} \in \mathcal{N}_{\mathrm{tr}} \cap \mathcal{M}^{\mathrm{tr}} = \mathcal{N}_{\mathrm{tr}} = \mathcal{D}(\mu_0)$. Note that $u_0 = \sum_k \mathrm{T}_{u_0} v_k = \mu_0(\mathrm{T}_{u_0})$. Therefore Lemma 3.1.8 yields the assertion in the bounded case since $\Delta = \mathrm{I}$.

3. The approximation in the unbounded case is the same as in the proof of Theorem 3.2.7. □

3.4 Type *III* Factors

3.4.1 Type III_λ Factors $(0 \leq \lambda < 1)$

Throughout this section $\mathcal{M}$ is a type III_λ $(0 \leq \lambda < 1)$ von Neumann factor acting on a Hilbert space $\mathcal{H}$ with a cyclic, separating vector $u_0 \in \mathcal{H}$. The aim of this section is to obtain a refinement of Theorem 2.2.8. To this end, we first establish a correspondence between vectors in $\mathcal{H}$ and operators affiliated with a type II_∞ subalgebra of $\mathcal{M}$ which allows a discrete decomposition of $\mathcal{M}$ (cf. Theorem 2.3.6). Then we can use the results of §3.3 and show how the modular objects can be expressed with the help of these operators.

We will first show that the III_λ case can essentially be reduced to a II_∞ problem.

Proposition 3.4.1. *Let* $u_0 \in \mathcal{H}$ *be a cyclic and separating vector for the* III_λ *factor* $\mathcal{M}$ $(0 < \lambda < 1)$. *Then there exist a* II_∞ *factor* $\mathcal{N}$ *acting on a Hilbert space* $\mathcal{K}$, *an automorphism* $\theta \in \mathrm{aut}(\mathcal{N})$, *and a cyclic and separating vector* $w_0 \in \mathcal{K}$ *for* $\mathcal{N}$ *such that the following conditions are satisfied:*

(i) $\mathcal{N}$ *has an n. s. f. trace* tr *such that* $\mathrm{tr} \circ \theta = \lambda\, \mathrm{tr}$,

(ii) $\mathcal{M}$ *is the crossed product of* $\mathcal{N}$ *by* θ, *i. e.* $\mathcal{M} = \mathcal{R}(\mathcal{N}, \theta)$,

(iii) $u_0 = w_0 \otimes x_0 \in \mathcal{K} \otimes l_2(\mathbb{Z}) = \mathcal{H}$ *where* $x_0 \in l_2(\mathbb{Z})$ *is the function* $x_0(h) = \delta_{0,h}$ $(h \in \mathbb{Z})$.

Proof. 1. Let ω be the vector state generated by u_0. Then there exists a generalized trace $\tau = \omega_{\mathrm{A}}$ $(\mathrm{A} \in \mathcal{M}^\omega,\ \lambda\mathrm{I} \leq \mathrm{A} < \mathrm{I})$ which commutes with ω [Str81, Theorem 30.7]. If we choose $\mathcal{N} = \mathcal{M}^\tau$ and $\theta = \mathrm{ad}\,\mathrm{U}_{|\mathcal{N}}$ where $\mathrm{U} \in \mathcal{M}$ is a unitary such that $\lambda\tau = \tau \circ \mathrm{ad}\,\mathrm{U}$ [Str81, § 29.5], $\mathcal{R}(\mathcal{N}, \theta)$ is a discrete decomposition of $\mathcal{M}$ (cf. Theorem 2.3.6 and Remark 2.3.7). In particular, there is an isomorphism $\pi : \mathcal{R}(\mathcal{N}, \theta) \to \mathcal{M}$ such that $\pi_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$.

2. Set $\mathcal{K} := [\mathcal{N}u_0]$ and let $\mathcal{N}$ act on $\mathcal{K}$, which is possible without loss of generality since u_0 is separating. Then $w_0 := u_0 \in \mathcal{K}$ is cyclic and separating for $\mathcal{N}$. A simple calculation implies that

$$\hat{u}_0 = [\delta_{0,n} w_0]_{n\in\mathbb{Z}} \in \bigoplus_{n\in\mathbb{Z}} \mathcal{K} = \mathcal{K} \otimes l_2(\mathbb{Z})$$

is cyclic and separating for $\mathcal{R}(\mathcal{N},\theta)$ acting on $\bigoplus_{n\in\mathbb{Z}} \mathcal{K}$. Let ω_0 be the vector state generated by $\hat{u}_0$. If we can show that $\omega_0 = \omega \circ \pi$, Theorem 2.1.10 implies that π is implemented by a unitary $\mathrm{V} : \bigoplus_{n\in\mathbb{Z}} \mathcal{K} \to \mathcal{H}$ such that $u_0 = \mathrm{V}\hat{u}_0$. To prove this, let

$$\mathrm{E}_0([\mathrm{U}(n-m)\mathrm{M}(n-m)]) := \mathrm{M}(0) \tag{3.4.1}$$

for all $[\mathrm{U}(n-m)\mathrm{M}(n-m)] \in \mathcal{R}(\mathcal{N},\theta)$ be the unique ω_0-invariant normal faithful conditional expectation from $\mathcal{R}(\mathcal{N},\theta)$ onto $\mathcal{N}$ (see Theorem 2.3.10). Then $(\pi \circ \mathrm{E}_0 \circ \pi^{-1})$ is ω-invariant and

$$\begin{aligned}\omega_0(\mathrm{M}) &= \omega_0(\mathrm{E}_0(\mathrm{M})) = \omega_0(\mathrm{M}(0)) = \omega(\pi(\mathrm{M}(0))) \\ &= \omega((\pi \circ \mathrm{E}_0 \circ \pi^{-1})(\pi(\mathrm{M}))) = \omega(\pi(\mathrm{M}))\end{aligned}$$

for all $\mathrm{M} = [\mathrm{U}(n-m)\mathrm{M}(n-m)] \in \mathcal{R}(\mathcal{N},\theta)$, which concludes the proof. □

The analogue of Proposition 3.4.1 for the type III_0 case is the following:

Proposition 3.4.2. *Let $u_0 \in \mathcal{H}$ be a cyclic and separating vector for the III_0 factor $\mathcal{M}$. Then there exist a II_∞ algebra $\mathcal{N}$ acting on a Hilbert space $\mathcal{K}$, an automorphism $\theta \in \mathrm{aut}(\mathcal{N})$, a cyclic and separating vector $w_0 \in \mathcal{K}$ for $\mathcal{N}$ such that the following conditions are satisfied:*

(i) $\mathcal{N}$ *has diffuse center and an n. s. f. trace* tr,

(ii) *there is an n. s. f. trace* tr *on* $\mathcal{N}$ *such that* $\mathrm{tr} \circ\theta \leq \lambda_0 \,\mathrm{tr}$ *for a* $0 < \lambda_0 < 1$,

(iii) $\mathcal{M}$ *is the crossed product of* $\mathcal{N}$ *by* θ, *i. e.* $\mathcal{M} = \mathcal{R}(\mathcal{N},\theta)$,

(iv) $u_0 = w_0 \otimes x_0 \in \mathcal{K} \otimes l_2(\mathbb{Z}) = \mathcal{H}$, *where* $x_0 \in l_2(\mathbb{Z})$ *is the function* $x_0(h) = \delta_{0,h}$ *($h \in \mathbb{Z}$).*

Proof. 1. Let ω be the vector state generated by u_0. Then there exists a lacunary weight of infinite multiplicity $\tau = \omega_{\mathrm{B}}$ ($\mathrm{B} \in \mathcal{M}^\omega$, $\mathrm{B} < \mathrm{I}$) which commutes with ω [Str81, Theorem 30.7]. We can again choose $\mathcal{N} = \mathcal{M}^\tau$ and $\theta = \mathrm{ad}\,\mathrm{U}_{|\mathcal{N}}$ with an appropriate chosen unitary $\mathrm{U} \in \mathcal{U}(\mathcal{M})$ (cf. [Str81, Lemma 30.2]) for the discrete decomposition of $\mathcal{M}$.

2. The same reasoning as in the second part of the proof of Proposition 3.4.1 leads to the assertion. □

Throughout the remainder of this subsection $\mathcal{N}$ is a type II_∞ algebra with cyclic and separating vector w_0, trace tr, and automorphism $\theta \in \mathrm{aut}(\mathcal{N})$ such that the properties of Proposition 3.4.1 or Proposition 3.4.2 are fulfilled. The results of §3.3 imply the existence of an orthogonal family $(v_k)_{k\in\mathbb{N}}$ in $\mathcal{K}$ which is cyclic for $\mathcal{N}$ and $\mathcal{N}'$ and a corresponding cyclic and separating generalized vector μ_0 for $\mathcal{N}$ generated by $(v_k)_{k\in\mathbb{N}}$. Furthermore, $\sum_k \langle \cdot v_k | v_k \rangle = \mathrm{tr}$ and there exists an invertible operator $\mathrm{T}_0 \in \mathcal{N}$ such that

$$w_0 = \sum_k \mathrm{T}_0 v_k \quad \text{and} \quad \mathrm{T}_0\mathrm{T}_0^* < \mathrm{I} \tag{3.4.2}$$

(see the proofs of Proposition 3.4.1 and Proposition 3.4.2, and Remark 3.3.7). Define $\hat{v}_k := [\delta_{0n} v_k]_{n\in\mathbb{Z}} \in \mathcal{H}$. Then

$$\tau = \widehat{\mathrm{tr}} = \mathrm{tr} \circ \mathrm{E}_0 = \sum_k \langle \cdot \hat{v}_k | \hat{v}_k \rangle,$$

where τ is the generalized trace (lacunary weight of infinite multiplicity) which exists according to Proposition 3.4.1 or Proposition 3.4.2, respectively, and E_0 is defined by (3.4.1). Moreover, setting

$$\hat{\mathrm{T}}_0 := [\delta_{n,m} \mathrm{T}_0], \tag{3.4.3}$$

we obtain $u_0 = \sum_k \hat{\mathrm{T}}_0 \hat{v}_k$.

Defining

$$\begin{aligned} \tilde{\mu}_0 : \mathcal{D}(\tilde{\mu}_0) \subset \mathcal{M} &\to \mathcal{H} \\ \mathrm{M} &\mapsto \tilde{\mu}_0(\mathrm{M}) := \sum_k \mathrm{M}\hat{v}_k \end{aligned} \tag{3.4.4a}$$

with domain

$$\mathcal{D}(\tilde{\mu}_0) = \mathcal{N}_\tau = \{\mathrm{M} \in \mathcal{M} : \tau(\mathrm{M}^*\mathrm{M}) = \sum_k \|\mathrm{M}\hat{v}_k\|^2 < \infty\}, \tag{3.4.4b}$$

$\tilde{\mu}_0$ is a cyclic generalized vector for $\mathcal{M}$ such that $\tilde{\mu}_0((\mathcal{D}(\tilde{\mu}_0)^* \cap \mathcal{D}(\tilde{\mu}_0))^2)$ is dense in $\mathcal{H}$ (cf. §3.3). Let further

$$\begin{aligned} \mathcal{D}(\tilde{\mu}_0^{\mathrm{c}}) &= \{\mathrm{M}' \in \mathcal{M}' : \exists v_{\mathrm{M}'} : \mathrm{M}'\tilde{\mu}_0(\mathrm{X}) = \mathrm{X}v_{\mathrm{M}'} \quad \forall \mathrm{X} \in \mathcal{D}(\tilde{\mu}_0)\} \\ \tilde{\mu}_0^{c}(\mathrm{M}') &= v_{\mathrm{M}'} \quad \forall \mathrm{M}' \in \mathcal{D}(\tilde{\mu}_0^{\mathrm{c}}) \end{aligned}$$

be the commutant of $\tilde{\mu}_0$. With the same calculations as in §3.3 we get

$$\begin{aligned} \mathcal{D}(\tilde{\mu}_0^{\mathrm{c}}) &= \{\mathrm{M}' \in \mathcal{M}' : \sum_k \left\|\mathrm{M}'\hat{v}_k\right\|^2 < \infty\} \\ \tilde{\mu}_0^{c}(\mathrm{M}') &= \sum_k \mathrm{M}'\hat{v}_k. \end{aligned}$$

Then $\tilde{\mu}_0^{\mathrm{c}}$ is a cyclic generalized vector for $\mathcal{M}'$ such that $\tilde{\mu}_0^{\mathrm{c}}((\mathcal{D}(\tilde{\mu}_0^{\mathrm{c}})^* \cap \mathcal{D}(\tilde{\mu}_0^{\mathrm{c}}))^2)$ is dense in $\mathcal{H}$. Hence, $\tilde{\mu}_0$ is a cyclic and separating generalized vector for $\mathcal{M}$ fulfilling the prerequisites of Lemma 3.1.5 and Theorem 3.1.6.

Proposition 3.4.3. *The modular operator corresponding to $\tilde{\mu}_0$ is*

$$\begin{aligned}\mathcal{D}(\Delta_{\tilde{\mu}_0}) &= \{u=[u_n]_n \in \mathcal{H} | \mathrm{U}(n)^* u_n \in \mathcal{D}(\mathrm{A_n}), \sum_n \|\mathrm{U}(n)\mathrm{A}_n\mathrm{U}(n)^* u_n\|^2 < \infty\} \\ \Delta_{\tilde{\mu}_0}([u_n]_n) &= [\mathrm{U}(n)\mathrm{A}_n\mathrm{U}(n)^* u_n]_n \quad \forall u=[u_n]_n \in \mathcal{D}(\Delta_{\tilde{\mu}_0})\end{aligned} \tag{3.4.5}$$

($\Delta_{\tilde{\mu}_0} = [\delta_{n,m}\mathrm{U}(n)\mathrm{A}_n\mathrm{U}(n)^*]_{n,m} = \bigoplus_{n\in\mathbb{Z}} \mathrm{U}(n)\mathrm{A}_n\mathrm{U}(n)^*$, for short) where A_n is the unique positive invertible operator affiliated with $\mathcal{Z}(\mathcal{N})$ such that $\mathrm{tr}\circ\theta^n = (\mathrm{tr})_{\mathrm{A}_n}$ ($n \in \mathbb{Z}$) (cf. Theorem 2.1.16 for the notation) and $\mathrm{ad}\,\mathrm{U}(n) = \theta^n$.

Remark 3.4.4. Note that Proposition 3.4.3 was proved in a slightly different context in [Sun87, Proposition 4.2.5].

Proof of Proposition 3.4.3. The proof is essentially a adaptation of the proof of Proposition 4.2.5 in [Sun87] for the different context used here. We sketch it for the sake of clearness.

Let $\mathrm{S}_{\tilde{\mu}_0}$ be the Tomita operator corresponding to $\tilde{\mu}_0$ defined by

$$\mathrm{S}_{\tilde{\mu}_0}\tilde{\mu}_0(\mathrm{M}) = \tilde{\mu}_0(\mathrm{M}^*) \quad \text{for } \mathrm{M} \in \mathcal{D}(\tilde{\mu}_0) \cap \mathcal{D}(\tilde{\mu}_0)^*.$$

Let further $\tilde{\Delta}$ denote the operator defined by (3.4.5). Define

$$\begin{aligned}D_0 := \{u=[u_n]_n \in \mathcal{H} | \mathrm{U}(n)^* u_n \in \mu_0(\mathcal{D}(\mu_0) \cap \mathcal{D}(\mu_{\mathrm{A_n}})) = \mu_0(\mathcal{N}_{\mathrm{tr}} \cap \mathcal{N}_{\mathrm{tr}_{\mathrm{A}_n^2}}), \\ u_n \neq 0 \text{ only for finitely many } n \in \mathbb{Z}\},\end{aligned} \tag{3.4.6}$$

where μ_{A_n} is the cyclic and separating generalized vector for $\mathcal{N}$ which defined by $\mu_{\mathrm{A}_n}(\cdot) = \mu_0(\mathrm{A}_n \cdot)$. It generates the n.s.f. trace $\mathrm{tr}_{\mathrm{A}_n^2}$ on $\mathcal{N}$. Then D_0 is a core for $\tilde{\Delta}$ since $\mu_0(\mathcal{N}_{\mathrm{tr}} \cap \mathcal{N}_{\mathrm{tr}_{\mathrm{A}_n^2}})$ is a core for A_n ($n \in \mathbb{Z}$) [Sun87, Lemma 4.2.3].

Let $u = [u_n]_n \in D_0$, i.e. there are $\mathrm{N}(n) \in \mathcal{N}_{\mathrm{tr}} \cap \mathcal{N}_{\mathrm{tr}_{\mathrm{A}_n^2}} \subset \mathcal{N}$ such that $\mathrm{U}(n)^* u_n = \mu_0(\mathrm{N}(n))$. [Sun87, Lemma 4.2.4] implies

$$\mathrm{N} := [\mathrm{U}(n-m)\mathrm{N}(n-m)] \in \mathcal{D}(\tilde{\mu}_0) \cap \mathcal{D}(\tilde{\mu}_0)^*.$$

Hence, $u = \tilde{\mu}_0(\mathrm{N}) \in \mathcal{D}(\tilde{\mathrm{S}})$.

Let further $\mathrm{M} \in \mathcal{D}(\tilde{\mu}_0) \cap \mathcal{D}(\tilde{\mu}_0)^* \subset \mathcal{M}$ with $\mathrm{M} = [\mathrm{U}(n-m)\mathrm{M}(n-m)]_{n,m}$. Then

$$\begin{aligned}\langle \mathrm{S}_{\tilde{\mu}_0}\tilde{\mu}_0(\mathrm{M}) | \mathrm{S}_{\tilde{\mu}_0}\tilde{\mu}_0(\mathrm{N}) \rangle &= \langle \tilde{\mu}_0(\mathrm{M}^*) | \tilde{\mu}_0(\mathrm{N}^*) \rangle \\ &= \langle \sum_k \mathrm{M}^* \hat{v}_k | \sum_k \mathrm{N}^* \hat{v}_k \rangle \\ &= \sum_n \sum_k \langle \mathrm{M}(-n)^* \mathrm{U}(n) v_k | \mathrm{N}(-n)^* \mathrm{U}(n) v_k \rangle \\ &= \sum_n \mathrm{tr}(\mathrm{U}(n)^* \mathrm{N}(-n) \mathrm{M}(-n)^* \mathrm{U}(n)) \\ &= \sum_n \mathrm{tr}(\mathrm{A}_{-n} \mathrm{N}(-n) \mathrm{M}(-n)^*) \\ &= \sum_n \mathrm{tr}(\mathrm{M}(n)^* \mathrm{A}_n \mathrm{N}(n)).\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_n \operatorname{tr}(\mathrm{M}(n)^*\mathrm{A}_n\mathrm{N}(n)) &= \sum_n\sum_k \langle \mathrm{A}_n\mathrm{N}(n)v_k|\mathrm{M}(n)v_k\rangle \\ &= \sum_n\sum_k \langle \mathrm{U}(n)\mathrm{A}_n\mathrm{U}(n)^*\mathrm{U}(n)\mathrm{N}(n)v_k|\mathrm{U}(n)\mathrm{M}(n)v_k\rangle \\ &= \langle \tilde{\Delta}\tilde{\mu}_0(\mathrm{N})|\tilde{\mu}_0(\mathrm{M})\rangle.\end{aligned}$$

Hence, $u = \tilde{\mu}_0(\mathrm{N}) \in \mathcal{D}(\mathrm{S}^*_{\tilde{\mu}_0}\mathrm{S}_{\tilde{\mu}_0}) = \mathcal{D}(\Delta_{\tilde{\mu}_0})$ and $\tilde{\Delta}u = \mathrm{S}^*_{\tilde{\mu}_0}\mathrm{S}_{\tilde{\mu}_0}u = \Delta_{\tilde{\mu}_0}u$. Thus $\Delta_{\tilde{\mu}_0}$ and $\tilde{\Delta}$ coincide on the core D_0 for $\tilde{\Delta}$. Since $\tilde{\Delta}$ and $\Delta_{\tilde{\mu}_0}$ are selfadjoint this implies $\tilde{\Delta} = \Delta_{\tilde{\mu}_0}$. □

Corollary 3.4.5. *Let* $\mathrm{H} \in \mathcal{M}^\tau$, *i. e.* H *has matrix* $[\delta_{n,m}\mathrm{B}]_{n,m}$ *with* $\mathrm{B} \in \mathcal{N}$. *Then*

$$\mathrm{J}_{\tilde{\mu}_0}\mathrm{H}\mathrm{J}_{\tilde{\mu}_0} = [\delta_{n,m}\mathrm{U}(n)\mathrm{JBJU}(n)^*]$$

where $\mathrm{J}_{\tilde{\mu}_0}$ *is the modular conjugation corresponding to* $\tilde{\mu}_0$ *and* J *is the modular conjugation corresponding to the cyclic and separating generalized vector* μ_0 *for* $\mathcal{N} = M^\tau$.

Proof. Let D_0 be the set defined by (3.4.6). D_0 is then a core for $\Delta^{1/2}_{\tilde{\mu}_0}$ [Sun87, Lemma 4.2.3]. Let $u \in \Delta^{1/2}_{\tilde{\mu}_0}D_0$, i. e. $u = \Delta^{1/2}_{\tilde{\mu}_0}[\mathrm{U}(n)\sum_k \mathrm{N}(n)v_k]_n$ where $\mathrm{N}(n) \in \mathcal{N}$ $(n \in \mathbb{Z})$ and $\mathrm{N}(n) \neq 0$ for finitely many $n \in \mathbb{Z}$. Then

$$\begin{aligned}\mathrm{J}_{\tilde{\mu}_0}\mathrm{H}\mathrm{J}_{\tilde{\mu}_0}u &= \mathrm{J}_{\tilde{\mu}_0}[\delta_{n,m}\mathrm{B}]_{n,m}\mathrm{S}_{\tilde{\mu}_0}\left[\sum_k \mathrm{U}(n)\mathrm{N}(n)v_k\right]_n \\ &= \mathrm{J}_{\tilde{\mu}_0}[\delta_{n,m}\mathrm{B}]_{n,m}\left[\sum_k \mathrm{N}(-n)^*\mathrm{U}(n)v_k\right]_n \qquad (3.4.7) \\ &= \mathrm{J}_{\tilde{\mu}_0}\left[\sum_k \mathrm{BN}(-n)^*\mathrm{U}(n)v_k\right]_n .\end{aligned}$$

Now

$$\begin{aligned}&\operatorname{tr}_{\mathrm{A}_n}(\mathrm{U}(n)^*\mathrm{N}(-n)\mathrm{B}^*\mathrm{U}(n)\mathrm{U}(n)^*\mathrm{BN}(-n)^*\mathrm{U}(n)) = \\ &\qquad = \operatorname{tr}(\mathrm{N}(-n)\mathrm{B}^*\mathrm{BN}(-n)^*) \leq \|\mathrm{B}\|^2\operatorname{tr}(\mathrm{N}(-n)^*\mathrm{N}(-n)) < \infty\end{aligned}$$

(cf. the definition of D_0), and therefore

$$\begin{aligned}&\sum_k \left\|\mathrm{A}_n^{1/2}\mathrm{U}(n)^*\mathrm{BN}(-n)^*\mathrm{U}(n)v_k\right\| = \\ &\qquad = \operatorname{tr}_{\mathrm{A}_n}(\mathrm{U}(n)^*\mathrm{N}(-n)\mathrm{B}^*\mathrm{U}(n)\mathrm{U}(n)^*\mathrm{BN}(-n)^*\mathrm{U}(n)) < \infty\end{aligned}$$

which implies $\sum_k \mathrm{U}(n)^*\mathrm{BN}(-n)^*\mathrm{U}(n)v_k \in \mathcal{D}(\mathrm{A}_\mathrm{n}^{1/2})$ and

$$\left[\sum_k \mathrm{BN}(-n)^*\mathrm{U}(n)v_k\right]_n \in \mathcal{D}(\Delta^{1/2}_{\tilde{\mu}_0}).$$

Hence, we can continue (3.4.7):

$$\begin{aligned}
\mathrm{J}_{\tilde{\mu}_0}\mathrm{H}\mathrm{J}_{\tilde{\mu}_0}u &= \Delta_{\tilde{\mu}_0}^{1/2}\mathrm{J}_{\tilde{\mu}_0}\Delta_{\tilde{\mu}_0}^{1/2}\left[\sum_k \mathrm{B}\mathrm{N}(-n)^*\mathrm{U}(n)v_k\right]_n \\
&= \Delta_{\tilde{\mu}_0}^{1/2}\left[\sum_k \mathrm{U}(n)\mathrm{N}(n)\mathrm{B}^*v_k\right]_n \\
&= \left[\mathrm{A}_n^{1/2}\sum_k \mathrm{U}(n)\mathrm{N}(n)\mathrm{B}^*v_k\right]_n \\
&= \left[\mathrm{A}_n^{1/2}\sum_k \mathrm{U}(n)\mathrm{J}\mathrm{B}\mathrm{N}(n)^*v_k\right]_n \\
&= \left[\mathrm{A}_n^{1/2}\sum_k \mathrm{U}(n)\mathrm{J}\mathrm{B}\mathrm{J}\mathrm{U}(n)^*\mathrm{U}(n)\mathrm{N}(n)v_k\right]_n \\
&= \left[\sum_k \mathrm{U}(n)\mathrm{J}\mathrm{B}\mathrm{J}\mathrm{U}(n)^*\mathrm{A}_n^{1/2}\mathrm{U}(n)\mathrm{N}(n)v_k\right]_n .
\end{aligned}$$

The latter holds since $\mathrm{A}_n^{1/2}\eta\mathcal{Z}(\mathcal{M}^\tau) = \mathcal{Z}(\mathcal{N})$. Moreover,

$$\begin{aligned}
\mathrm{J}_{\tilde{\mu}_0}\mathrm{H}\mathrm{J}_{\tilde{\mu}_0}u &= \left[\delta_{n,m}\mathrm{U}(n)\mathrm{J}\mathrm{B}\mathrm{J}\mathrm{U}(n)^*\right]_n \Delta_{\tilde{\mu}_0}^{1/2}\left[\mathrm{U}(n)\sum_k \mathrm{N}(n)v_k\right]_n \\
&= \left[\delta_{n,m}\mathrm{U}(n)\mathrm{J}\mathrm{B}\mathrm{J}\mathrm{U}(n)^*\right]_n u.
\end{aligned}$$

Since $\Delta_{\tilde{\mu}_0}^{1/2}D_0$ is dense in $\mathcal{H}$ this completes the proof. □

Theorem 3.4.6. *Let $\mathcal{M}$ be a type III_λ factor ($0 \leq \lambda < 1$) and u_0 be a cyclic and separating vector for $\mathcal{M}$. Let further $\hat{\mathrm{T}}_0 \in \mathcal{N}$ be the invertible operator corresponding to u_0 (see* (3.4.2) *and* (3.4.3)*), where $(\mathcal{N}, \theta, \mathrm{tr})$ is the discrete decomposition of $\mathcal{M}$ corresponding to u_0 (see Proposition 3.4.1 and Proposition 3.4.2). Let $\hat{\mathrm{T}}_0 = \hat{\mathrm{H}}_0^{1/2}\hat{\mathrm{V}}$ be the polar decomposition of $\hat{\mathrm{T}}_0$.*

Then the modular objects (Δ_0, J_0) of $(\mathcal{M}, u_0)$ are

$$\mathrm{J}_0 = \mathrm{J}_{\tilde{\mu}_0}\hat{\mathrm{V}}^*\mathrm{J}_{\tilde{\mu}_0}\hat{\mathrm{V}}\mathrm{J}_{\tilde{\mu}_0},$$

where $\mathrm{J}_{\tilde{\mu}_0}$ is the conjugation corresponding to the cyclic and separating generalized vector $\tilde{\mu}_0$ which generates the dual weight of tr*, and*

$$\Delta_0 = \hat{\mathrm{H}}_0\mathrm{J}_0\hat{\mathrm{H}}_0^{-1}\mathrm{J}_0\Delta_{\tilde{\mu}_0},$$

where $\Delta_{\tilde{\mu}_0}$ is the modular operator corresponding to $\tilde{\mu}_0$ (see Proposition 3.4.3).

Proof. The proof is exactly the same as that of Theorem 3.3.9. □

Remark 3.4.7. 1. In matrix notation we can write Δ_0 as

$$\Delta_0 = [\delta_{n,m}\mathrm{H}_0\mathrm{U}(n)\mathrm{J}\mathrm{H}_0^{-1}\mathrm{J}\mathrm{U}(n)^*\mathrm{A}_n] = \bigoplus_n \mathrm{H}_0\mathrm{U}(n)\mathrm{J}\mathrm{H}_0^{-1}\mathrm{J}\mathrm{U}(n)^*\mathrm{A}_n$$

(cf. Corollary 3.4.5).

2. For a similar result compare with [Iso95].

3.4.2 Remarks on Modular Operators for Type III_1 Factors

In this section we consider modular operators for type III_1 factors. As mentioned in Remark 2.3.7 there is in general no discrete decomposition for type III_1 factors unlike in the type III_λ case [Con74]. Therefore we can not use the techniques of §3.4.1. Nevertheless, the following lemma provides a similar result at least in the hyperfinite case (a von Neumann algebra is called *hyperfinite* if it is the weak closure of the union of finite dimensional von Neumann algebras).

Lemma 3.4.8. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two von Neumann algebras and α, β two actions of the (discrete) groups G and H on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then the tensor product $\mathcal{R} = \mathcal{R}(\mathcal{M}_1, \alpha) \otimes \mathcal{R}(\mathcal{M}_2, \beta)$ of the crossed products of $\mathcal{M}_1$ by α and $\mathcal{M}_2$ by β is the crossed product of $\mathcal{M}_1 \otimes \mathcal{M}_2$ by the direct product of α and β, i. e. with $\gamma_{(g,h)} := \alpha_g \otimes \beta_h$ for all $g \in G$ and $h \in H$ we have $\mathcal{R} = \mathcal{R}(\mathcal{M}_1 \otimes \mathcal{M}_2, \gamma)$.*

Proof. For the proof let α be implemented by a unitary group $\mathrm{V} \in \mathcal{U}(\mathcal{H}_1)$ and β by $\mathrm{W} \in \mathcal{U}(\mathcal{H}_2)$, i.e. $\alpha_g = \mathrm{ad}\,\mathrm{V}_g$ for all $g \in G$ and $\beta_h = \mathrm{ad}\,\mathrm{W}_h$ for all $h \in G$. Let further $\mathcal{R} := \mathcal{R}(\mathcal{M}_1, \alpha) \otimes \mathcal{R}(\mathcal{M}_2, \beta)$ and $\tilde{\mathcal{R}} := \mathcal{R}(\mathcal{M}_1 \otimes \mathcal{M}_2, \gamma)$ where $\gamma_{(g,h)} := \alpha_g \otimes \beta_h = \mathrm{ad}\,\mathrm{V}_g \otimes \mathrm{W}_h$ for all $(g,h) \in G \times H$ is an action of $G \times H$ on $\mathcal{M}_1 \otimes \mathcal{M}_2$.

Now $\mathcal{R}$ acts on $\mathcal{H}_1 \otimes l_2(G) \otimes \mathcal{H}_2 \otimes l_2(H)$ and is is generated by the operators $\mathrm{A} \otimes \mathrm{I} \otimes \mathrm{B} \otimes \mathrm{I}$ and $\mathrm{V}_g \otimes l_g \otimes \mathrm{W}_h \otimes l_h$ ($\mathrm{A} \in \mathcal{M}_1$, $\mathrm{B} \in \mathcal{M}_2$, $g \in G$, $h \in H$), whereas $\tilde{\mathcal{R}}$ acts on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes l_2(G \times H)$ and is generated by $\mathrm{A} \otimes \mathrm{B} \otimes \mathrm{I}$ and $\mathrm{V}_g \otimes \mathrm{W}_h \otimes l_{(g,h)}$ ($\mathrm{A} \in \mathcal{M}_1$, $\mathrm{B} \in \mathcal{M}_2$, $(g,h) \in G \times H$) (cf. Theorem 2.2.1 and Definition 2.2.4). Since $l_2(G \times H) = l_2(G) \otimes l_2(H)$ and $l_{(g,h)} = l_g \otimes l_h$ we can define an isomorphism σ from $\mathcal{R}$ onto $\tilde{\mathcal{R}}$ by $\sigma(\mathrm{A} \otimes \mathrm{I} \otimes \mathrm{B} \otimes \mathrm{I}) = \mathrm{A} \otimes \mathrm{B} \otimes \mathrm{I}$ and $\sigma(\mathrm{V}_g \otimes l_g \otimes \mathrm{W}_h \otimes l_h) = \mathrm{V}_g \otimes \mathrm{W}_h \otimes l_{(g,h)}$. □

Corollary 3.4.9. *Suppose in the situation of Lemma 3.4.8 that $\mathcal{M}_1$ and $\mathcal{M}_2$ are semifinite von Neumann algebras with traces tr_1 and tr_2. Assume that there exist positive numbers a_g and b_h such that $\mathrm{tr}_1 \circ \alpha_g = a_g \,\mathrm{tr}_1$ for all $g \in G$ and $\mathrm{tr}_2 \circ \beta_h = b_h \,\mathrm{tr}_2$ for all $h \in H$. Then*

$$(\mathrm{tr}_1 \otimes \mathrm{tr}_2) \circ \gamma_{(g,h)} = a_g b_h (\mathrm{tr}_1 \otimes \mathrm{tr}_2)$$

for all $(g,h) \in G \times H$.

Proof. This can be shown by an elementary calculation. □

It is known that the tensor product of a type III_{λ_1} factor and a type III_{λ_2} factor ($0 < \lambda_1, \lambda_2 < 1$) is a type III_1 factor if and only if $\ln(\lambda_1)/\ln(\lambda_2) \notin \mathbb{Q}$ (see e.g. Lemma A.4.1). Since every type III_λ factor ($0 < \lambda < 1$) possesses a discrete decomposition as a crossed product of a type II_∞ factor and an action of $\mathbb{Z}$ (cf. Theorem 2.3.6), Lemma 3.4.8 implies that such III_1 factors are the discrete crossed product of a type II_∞ factor by an action of $\mathbb{Z}^2$ (note that the tensor product of two II_∞ factors is a II_∞ factor, cf. Table 2.1).

Let now $\mathcal{M}$ be a type III_1 factor which is the tensor product of a type III_{λ_1} and a type III_{λ_2} factor ($0 < \lambda_1, \lambda_2 < 1$, $\ln(\lambda_1)/\ln(\lambda_2) \notin \mathbb{Q}$). $\mathcal{M}$ is hence the crossed product $\mathcal{R}(\mathcal{N}, \theta)$ of a type II_∞ factor $\mathcal{N}$ and an action θ of $\mathbb{Z}^2$ on

$\mathcal{N}$, according to Lemma 3.4.8. By Corollary 3.4.9, we have $\operatorname{tr} \circ \theta_{n,m} = \lambda_1^n \lambda_2^m \operatorname{tr}$ for $(n,m) \in \mathbb{Z}^2$. The hyperfinite factor of type III_1 fulfils this condition, for instance, since it is the tensor product of two hyperfinite III_λ factors.

Suppose that $\mathcal{N}$ acts on a Hilbert space $\mathcal{K}$ and that it possesses a cyclic and separating vector $w_0 \in \mathcal{K}$ and a trace tr. Assume further that θ is unitarily implemented, i. e. $\theta_{(n,m)} = \operatorname{ad} \mathrm{U}(n,m)$ with unitaries $\mathrm{U}(n,m) \in \mathcal{U}(\mathcal{K})$ for all $(n,m) \in \mathbb{Z}^2$. The results of §3.3 imply the existence of an orthogonal family $(v_k)_{k \in \mathbb{N}}$ in $\mathcal{K}$ which is cyclic for $\mathcal{N}$ and $\mathcal{N}'$. Furthermore, $\sum_k \langle \cdot v_k | v_k \rangle = \operatorname{tr}$ and there exists an invertible operator $\mathrm{T}_0 \eta \mathcal{N}$ such that $w_0 = \sum_k \mathrm{T}_0 v_k$ and $\operatorname{tr}(\mathrm{T}_0 \mathrm{T}_0^*) < \infty$ (see Lemma 3.3.5). We can now proceed as in §3.4.1 and obtain

Theorem 3.4.10. *Let $\mathcal{M}$ be a type III_1 factor and u_0 be a cyclic and separating vector for $\mathcal{M}$ which is the dual vector of the cyclic and separating vector w_0 for $\mathcal{N}$ where $(\mathcal{N}, \theta, \operatorname{tr})$ is the discrete decomposition of $\mathcal{M}$ described above such that $\operatorname{tr} \circ \theta_{n,m} = \lambda_1^n \lambda_2^m \operatorname{tr}$ for $(n,m) \in \mathbb{Z}^2$. Let further $\hat{\mathrm{T}}_0 \in \mathcal{N}$ be the invertible operator corresponding to u_0, and let $\hat{\mathrm{T}}_0 = \hat{\mathrm{H}}_0^{1/2} \hat{\mathrm{V}}$ be its polar decomposition. Then the modular objects (Δ_0, J_0) of $(\mathcal{M}, u_0)$ are*

$$\mathrm{J}_0 = \mathrm{J}_{\tilde{\mu}_0} \hat{\mathrm{V}}^* \mathrm{J}_{\tilde{\mu}_0} \hat{\mathrm{V}} \mathrm{J}_{\tilde{\mu}_0},$$

where $\mathrm{J}_{\tilde{\mu}_0}$ is the conjugation corresponding to the cyclic and separating generalized vector $\tilde{\mu}_0$ which generates the dual weight of tr, *and*

$$\Delta_0 = \hat{\mathrm{H}}_0 \mathrm{J}_0 \hat{\mathrm{H}}_0^{-1} \mathrm{J}_0 \Delta_{\tilde{\mu}_0}, \tag{3.4.8}$$

where $\Delta_{\tilde{\mu}_0} = \bigoplus_{(n,m) \in \mathbb{Z}^2} \lambda_1^n \lambda_2^m$ is the modular operator corresponding to $\tilde{\mu}_0$.

Proof. The proof is exactly the same as those of Theorem 3.3.9 and Theorem 3.4.6. □

Remark 3.4.11.

1. Since the operator $\Delta_{\tilde{\mu}_0}$ depends on the constants λ_1 and λ_2 the construction presented here depends highly on the composition of the type III_1 factor as a tensor product of III_λ factors and thus on the action of $\mathbb{Z}^2$ on $\mathcal{N}$. This is a difference to the type III_λ case $(0 \leq \lambda < 1)$ where all the discrete decompositions are equivalent. For instance, the hyperfinite type III_1 factor allows any decomposition as a tensor product of a type III_{λ_1} factor and a type III_{λ_2} factor provided $\ln(\lambda_1)/\ln(\lambda_2) \notin \mathbb{Q}$. Hence, every $\Delta_{\tilde{\mu}_0} = \bigoplus_{n,m \in \mathbb{Z}} \lambda_1^n \lambda_2^m$ can occur.

2. In contrast to the type III_λ case $(0 \leq \lambda < 1)$ there is no result analogous to Proposition 3.4.1 and Proposition 3.4.2. Therefore, Theorem 3.4.10 does not furnish the form of the modular objects for all cyclic and separating vectors for a (hyperfinite) type III_1 factor, as the following observation implies:

   (3.4.8) states that the modular operators considered are the direct sum of operators which are the product of an operator affiliated with a type II_∞ von Neumann algebra and an operator affiliated with its commutant. In §4.1 we will consider such products and will prove in Lemma 4.1.6 that the eigenvalues of such products have always infinite multiplicity if the algebra has no minimal projections. Type II_∞ algebras fulfil this condition. In particular, the eigenvalue 1 has infinite multiplicity. Examples in quantum field theory show however that there are modular operators for type III_1 factors which have only one eigenvector for the eigenvalue 1 (see Remark 2.4.3 and Remark 4.1.8).

3. Note again that Connes proved in [Con74] that there is no discrete decomposition for a general type III_1 factor. Hence, not all type III_1 factors can be written as a tensor product of type III_λ factors.

Chapter 4

Spectral Theory of Modular Operators

This chapter contains the systematic study of some spectral properties of modular operators corresponding to cyclic and separating vectors. To this end, we we will use the results of Chapter 3. For their greater technical simplicity we will restrict our attention to the so-called diagonalizable vectors. Up to now, it is not clear whether or nor there is an extension of the theory to the general case.

The main results of this chapter, and of this thesis as well, are Theorem 4.2.12, Theorem 4.3.12, and Theorem 4.4.7 which provide necessary and sufficient conditions for a positive operator to be a modular operator corresponding to a diagonalizable vector for a type I algebra, a type II algebra, or a type III_λ factor ($0 \leq \lambda < 1$), respectively.

This chapter is organized in four sections. The first section deals with the spectrum of products of two commuting operators where the first is affiliated with a von Neumann algebra $\mathcal{M}$ and the second is affiliated with the commutant $\mathcal{M}'$. These results are interesting in their own right and useful for the following as well since modular operators are composed of such products. The remaining sections contain the spectral theory of modular operators for the different types.

The results of this chapter are essentially new. Corollary 4.1.7 is a slight generalization of a result by Araki (cf. [Ara72]). Some aspects of the spectral theory for type I factors were considered previously by Wollenberg (cf. [Wol98] and [BW01]).

4.1 Remarks on the Product of Commuting Operators

In the previous chapter we showed that every modular operator for a semifinite von Neumann algebra $\mathcal{M}$ is the product of two positive commuting operators. More precisely, the first of these operators is affiliated with $\mathcal{M}$ and the second is affiliated with the commutant $\mathcal{M}'$ (cf. Theorem 3.3.9). Furthermore, if $\mathcal{M}$ is is a type III_λ factor ($0 \leq \lambda < 1$) the modular operators are direct sums of such products (cf. Remark 3.4.7). This observation motivates a more detailed investigation of the spectrum of such products.

Note first that the product of two commuting closed operators is closable (see e. g. [KR83, Theorem 5.6.15]). It therefore makes sense to speak of the (closed) product of two such operators meaning the closure of the product.

Let $\mathrm{H}\eta\mathcal{M}$, $\mathrm{H}'\eta\mathcal{M}'$ be two self-adjoint operators with spectral measures E and E'. Then there exists a joint spectral measure F of the two operators on the Borel-σ-algebra of $\mathbb{R}^2$ such that the restriction of this measure to the first and second component is the spectral measure of H and H', respectively, i. e.

$$\mathrm{F}(M_1 \times M_2) = \mathrm{E}(M_1)\mathrm{E}'(M_2) \tag{4.1.1}$$

for all Borel sets M_1, M_2 in $\mathbb{R}$ (see e. g. [BS87, § 6.5]).

Whereas the joint spectrum $\sigma(\mathrm{H},\mathrm{H}')$ of H and H' in general only fulfils

$$\sigma(\mathrm{H},\mathrm{H}') = \mathrm{supp}(\mathrm{F}) \subset \mathrm{supp}(\mathrm{E}) \times \mathrm{supp}(\mathrm{E}') = \sigma(\mathrm{H}) \times \sigma(\mathrm{H}')$$

(see e. g. [BS87, § 6.5]), we have the following result in our situation if $\mathcal{M}$ is a factor:

Lemma 4.1.1. *Let $\mathcal{M}$ be a von Neumann factor, and let $\mathrm{H}\eta\mathcal{M}$ and $\mathrm{H}'\eta\mathcal{M}'$ be two selfadjoint operators with spectral measures E and E'. Suppose that F is the joint spectral measure of H and H'. Then*

$$\sigma(\mathrm{H},\mathrm{H}') = \mathrm{supp}(\mathrm{F}) = \mathrm{supp}(\mathrm{E}) \times \mathrm{supp}(\mathrm{E}') = \sigma(\mathrm{H}) \times \sigma(\mathrm{H}').$$

Before we can prove this lemma we cite the following

Theorem 4.1.2. *Let $\mathcal{M}$ be a von Neumann algebra. Then $\mathrm{AA}' = 0$ with $\mathrm{A} \in \mathcal{M}$ and $\mathrm{A}' \in \mathcal{M}'$ if and only if $\mathrm{C_A C_{A'}} = 0$, where the central projections $\mathrm{C_A}, \mathrm{C_{A'}} \in \mathcal{Z}(\mathcal{M})$ are the central carriers of A and A'.*

A proof can be found in [KR83, Theorem 5.5.4].

Proof of Lemma 4.1.1. Let $M_1, M_2 \subset \mathbb{R}$ be Borel sets. Since $\mathcal{M}$ is a factor Theorem 4.1.2 implies that $\mathrm{E}(M_1)\mathrm{E}'(M_2) \neq 0$ if and only if $\mathrm{E}(M_1) \neq 0 \neq \mathrm{E}'(M_2)$.

We now have to prove $\mathrm{supp}(\mathrm{F}) \supset \mathrm{supp}(\mathrm{E}) \times \mathrm{supp}(\mathrm{E}')$. To this end, let $(\lambda_1, \lambda_2) \in \mathrm{supp}(\mathrm{E}) \times \mathrm{supp}(\mathrm{E}')$ and let $M_1, M_2 \subset \mathbb{R}$ be two arbitrary intervals with $\lambda_1 \in M_1$ and $\lambda_2 \in M_2$. Then $\mathrm{F}(M_1 \times M_2) = \mathrm{E}(M_1)\mathrm{E}'(M_2) \neq 0$ since $\lambda_1 \in \mathrm{supp}(\mathrm{E})$ and $\lambda_2 \in \mathrm{supp}(\mathrm{E}')$. Because the rectangles $M_1 \times M_2$ form a basis of the Borel-σ-algebra of $\mathbb{R}^2$ this implies $(\lambda_1, \lambda_2) \in \mathrm{supp}(\mathrm{F})$. □

Corollary 4.1.3. *Let $\mathcal{M}$ be a von Neumann factor, and let $\mathrm{H}\eta\mathcal{M}$ and $\mathrm{H}'\eta\mathcal{M}'$ be two selfadjoint operators. Let further $\Delta = \mathrm{HH}'$ be the (closed) product of H and H'. Then the spectrum $\sigma(\Delta)$ of Δ is*

$$\sigma(\Delta) = \overline{\sigma(\mathrm{H})\sigma(\mathrm{H}')} = \overline{\{\lambda_1\lambda_2 | \lambda_1 \in \sigma(\mathrm{H}), \lambda_2 \in \sigma(\mathrm{H}')\}}.$$

Proof. Note that $\Delta = \int_{\mathbb{R}^2} \lambda_1\lambda_2 d\mathrm{F}_{\lambda_1,\lambda_2}$, and $\sigma(f(\mathrm{H},\mathrm{H}')) = \overline{f(\sigma(\mathrm{H},\mathrm{H}'))}$ for all continuous functions f (see [BS87, § 6.6]). □

In the non-factor case the problem is more subtle:

Lemma 4.1.4. *Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathrm{H}\eta\mathcal{M}$ and $\mathrm{H}'\eta\mathcal{M}'$ be two selfadjoint operators with spectral measures E and E'. The irregular spectrum σ_{irr} is defined by*

$$\begin{aligned}\sigma_{irr} := \{&(\lambda_1, \lambda_2) \in \mathrm{supp}(\mathrm{E}) \times \mathrm{supp}(\mathrm{E}')| \\ &\exists M_1, M_2 \subset \mathbb{R} \textit{ intervals with } \lambda_1 \in M_1, \lambda_2 \in M_2 \textit{ and } \mathrm{C}_{\mathrm{E}(M_1)}\mathrm{C}_{\mathrm{E}'(M_2)} = 0\}\end{aligned}$$

where $\mathrm{C}_{\mathrm{E}(M_i)}$ is the central carrier of $\mathrm{E}(M_i)$ ($i = 1, 2$). Then the support $\mathrm{supp}(\mathrm{F})$ of the joint spectral measure F of H and H' is

$$\mathrm{supp}(\mathrm{F}) = \mathrm{supp}(\mathrm{E}) \times \mathrm{supp}(\mathrm{E}') \setminus \sigma_{irr}.$$

Proof. Let $(\lambda_1, \lambda_2) \in \mathrm{supp}(\mathrm{F})$, i.e. $0 \neq \mathrm{F}(M_1 \times M_2) = \mathrm{E}(M_1)\mathrm{E}'(M_2)$ holds for all intervals $M_1, M_2 \subset \mathbb{R}$ with $\lambda_1 \in M_1$ and $\lambda_2 \in M_2$. Theorem 4.1.2 now implies $\mathrm{E}(M_1) \neq 0$, $\mathrm{E}(M_1) \neq 0$ and $\mathrm{C}_{\mathrm{E}(M_1)}\mathrm{C}_{\mathrm{E}'(M_2)} \neq 0$ and therefore $(\lambda_1, \lambda_2) \in \mathrm{supp}(\mathrm{E}) \times \mathrm{supp}(\mathrm{E}') \setminus \sigma_{irr}$.

Let now $(\lambda_1, \lambda_2) \in \mathrm{supp}(\mathrm{E}) \times \mathrm{supp}(\mathrm{E}') \setminus \sigma_{irr}$, i.e. $\mathrm{E}(M_1) \neq 0 \neq \mathrm{E}'(M_2)$ and $\mathrm{C}_{\mathrm{E}(M_1)}\mathrm{C}_{\mathrm{E}'(M_2)} \neq 0$ hold for all intervals $M_1, M_2 \subset \mathbb{R}$ with $\lambda_1 \in M_1$ and $\lambda_2 \in M_2$. Then $\mathrm{F}(M_1 \times M_2) = \mathrm{E}(M_1)\mathrm{E}'(M_2) \neq 0$ (see again Theorem 4.1.2) and $(\lambda_1, \lambda_2) \in \mathrm{supp}(\mathrm{F})$. □

Corollary 4.1.5. *Let $\Delta = \mathrm{HH}'$ be the (closed) product of H and H'. Then the spectrum $\sigma(\Delta)$ of Δ is the closure of the following set:*

$$\begin{aligned}&\sigma(\mathrm{H})\sigma(\mathrm{H}') \setminus \{\mu \in \mathbb{R}| \\ &\quad \textit{for all } \lambda_1 \in \sigma(\mathrm{H}), \lambda_2 \in \sigma(\mathrm{H}') \textit{ such that } \mu = \lambda_1\lambda_2 \textit{ holds } (\lambda_1, \lambda_2) \in \sigma_{irr}\}.\end{aligned}$$

Lemma 4.1.6. *Let $\mathcal{M}$ be a von Neumann algebra with no non-zero minimal projections. Let further $\mathrm{H}\eta\mathcal{M}$ and $\mathrm{H}'\eta\mathcal{M}'$ be two selfadjoint operators and let $\Delta = \mathrm{HH}'$ be their product. Then every eigenvalue of Δ has infinite multiplicity.*

Proof. We denote the spectral measures of H and H' by E and E', respectively. Let further F be the joint spectral measure of H and H'. Then the spectral measure G of Δ is

$$\mathrm{G}(M) = \int\limits_{\{\lambda_1\lambda_2 \in M\}} d\mathrm{F}((\lambda_1, \lambda_2))$$

for all Borel sets $M \subset \mathbb{R}$. Let now ν be an eigenvalue of Δ, i.e. $\mathrm{G}(\{\nu\}) \neq 0$. We can then distinguish two cases:

1. Assume that there exists a pair $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that $\lambda_1\lambda_2 = \nu$ and

   $$0 \neq \mathrm{F}(\{(\lambda_1, \lambda_2)\}) = \mathrm{E}(\{\lambda_1\})\mathrm{E}'(\{\lambda_2\}).$$

   Set $\mathrm{E}_1 := \mathrm{E}(\{\lambda_1\})$, $\mathrm{E}_2' := \mathrm{E}'(\{\lambda_2\})$, and $\mathrm{P} := \mathrm{C}_{\mathrm{E}_1}\mathrm{C}_{\mathrm{E}_2'}$. Then $\mathrm{E}_1 \neq 0$, $\mathrm{E}_2' \neq 0$ and $\mathrm{P} \neq 0$ (cf. the proof of Lemma 4.1.4). Since $\mathcal{M}$ has no minimal projection there exists a projection $Q_1 \in \mathcal{M}$ such that $0 < Q_1 < E_1P \leq P$. We claim that $Q_1E_2^{'} < E_1E_2^{'}$. Indeed, Theorem 4.1.2 implies that $C_{E_1-Q_1}P \neq 0$ if and only if $(E_1 - Q_1)P = E_1P - Q_1 \neq 0$. The latter is implied by the properties of Q_1 such that we obtain

$$C_{E_1P-Q_1}C_{E_2^{'}} = C_{E_1-Q_1}P \neq 0$$

which is equivalent to $Q_1E_2^{'} < E_1PE_2^{'} = E_1E_2^{'}$.

Repeating this argument, we can construct inductively a descending sequence of non-zero projections $(Q_n)_{n\in\mathbb{N}}$ such that

$$0 \neq Q_nE_2^{'} < Q_mE_2^{'} < E_1E_2^{'} = F(\{(\lambda_1, \lambda_2)\})$$

for $n > m$. We thus have proved that $G(\{\nu\}) \geq F(\{(\lambda_1, \lambda_2)\})$ is infinite dimensional.

2. Assume now that $0 = F(\{(\lambda_1, \lambda_2)\})$ holds for all $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ with $\lambda_1\lambda_2 = \nu$. Set $K := \operatorname{supp}(F) \cap \{\lambda_1\lambda_2 = \nu\}$. We will prove indirectly that there exists a closed subset M_1 of K with $0 \neq F(M_1) < F(K)$. Accordingly, we suppose that

$$F(M) = \begin{cases} F(K) \text{ or} \\ 0 \end{cases} \tag{4.1.2}$$

holds for all closed subsets M of K.

Let now $(K_n^1)_{n\in N_1}$ be a covering of K with closed rectangles such that $K_n^1 \cap K_m^1$ contains at most one point for $n \neq m$ and diameter $d(K_n^1) = 1$ $(n \in N_1)$. All the sets K_n^1 are compact and $F(K) = \sum_n F(K_n^1 \cap K)$. (4.1.2) now yields exactly one $n_1 \in \mathbb{N}$ such that $F(K_{n_1}^1 \cap K) = F(K)$.

Repeat the argument with $K_{n_1}^1 \cap K$ replaced by K and closed rectangles K_n^2 with $d(K_n^2) = 1/2$ $(n \in N_2)$ and so on. Consequently, we get inductively a descending sequence $(K_{n_m}^m \cap K)_{m\in\mathbb{N}}$ of compact sets with $F(K) = F(K_{n_m}^m \cap K)$ and $d(K_{n_m}^m) = 1/m$ for $m \in \mathbb{N}$. This implies the existence of a pair $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that $\lambda_1\lambda_2 = \nu$, $\bigcap_m K_{n_m}^m \cap K = \{(\lambda_1, \lambda_2)\}$, and $F(\{(\lambda_1, \lambda_2)\}) = F(K) \neq 0$. This contradicts our initial assumption (4.1.2). Hence, there is a closed set M_1 with $0 \neq F(M_1) < F(K)$.

Repeating this argument we get an infinite sequence of closed sets $M_n \subset K$ such that $0 \neq F(M_n) < F(M_m)$ for $n > m$. We have thus proved that $G(\{\nu\}) = F(K)$ has a descending sequence of infinitely many non-zero subprojections, i. e. it is infinite dimensional. $\square$

We now prove the following extension of a result first obtained by Araki in [Ara72]:

Corollary 4.1.7. *Let $\mathcal{M}$ be a direct sum of semifinite von Neumann algebras and type III factors, and let Δ be a modular operator for $\mathcal{M}$ corresponding to a cyclic and separating vector. Suppose that the eigenvalue 1 of Δ has finite multiplicity. Then $\mathcal{M}$ is the direct sum of finite type I algebras with finite dimensional centers and of type III_1 factors.*

Proof. Note first that the modular operator corresponding to a direct sum of cyclic and separating vectors $\oplus u_n$ is the direct sum of the modular operators corresponding to the vectors u_n.

Assume now that $\mathcal{M}$ has no central summand of type III_1. Then Δ is the direct sum of products of commuting operators where the first factor of each summand is affiliated with a semifinite von Neumann algebra and the second factor is affiliated with the commutant (see Theorem 3.3.9 and Theorem 3.4.6). Since 1 is an eigenvalue of Δ with finite multiplicity, $\mathcal{M}$ has a minimal projection according to Lemma 4.1.6. This implies that $\mathcal{M}$ has a central summand of type I with discrete center. If there was another direct summand which was not of type I with discrete center then Δ would have eigenvalue 1 with infinite multiplicity according to Lemma 4.1.6. This proves that $\mathcal{M}$ is the direct sum of type I algebras with discrete centers.

If there was a central summand of type I_∞ or with infinite dimensional center then the eigenvalue 1 would have infinite multiplicity according to Lemma 4.2.6. This is again a contradiction. Hence, $\mathcal{M}$ is the direct sum of finite type I algebras with finite dimensional centers. □

Remark 4.1.8.

1. Further analysis of the results of §4.2 also yields the following estimation of the dimension of the Hilbert space $\mathcal{H}$ on which $\mathcal{M}$ acts: Assume that $\mathcal{M}$ has no central portion of type III_1 and let $n \in \mathbb{N}$ be the multiplicity of the eigenvalue 1 of Δ. Then $\dim \mathcal{H} \leq n^2$ (cf. [Ara72]).

2. Araki proved his result under the assumption that 1 is an isolated eigenvalue. If the algebra has no central portion of type III_1 this property follows from the assumption that the eigenvalue 1 has finite multiplicity since operators acting on finite dimensional Hilbert spaces always have only isolated eigenvalues. In the type III_1 case the eigenvalue 1 is never isolated since the type III_1 factors are characterized by the property that the spectrum of all modular operators is the whole positive real line (cf. Definition 2.3.2).

3. In the type III_1 case all multiplicities of the eigenvalue 1 really occur. Remark 2.4.3 states that the multiplicity 1 can occur. Denote this factor by $\mathcal{M}$. In the next section we will prove that there is a type I_n factor $\mathcal{N}_n$ which has a modular operator such that the eigenvalue 1 has multiplicity n for all $n \in \mathbb{N}$. The tensor-product $\mathcal{M} \otimes \mathcal{N}_n$ is then a type III_1 factor which has a modular operator such that 1 has multiplicity n.

4.2 Type I Algebras

In this section we investigate the spectrum of modular operators for type I_n and type I_∞ algebras. The results obtained in this section will turn out to be quite satisfactory. For type I algebras it is possible to characterize all modular operators corresponding to cyclic and separating vectors by some of their spectral properties. This section falls into two parts. In the first we will investigate the spectrum of modular operators for type I algebras in detail. In the second part we will prove that the spectral properties obtained so far are not only necessary but also sufficient for a positive invertible operator being a modular operator.

Throughout this section $\mathcal{M}$ is a type I_n or type I_∞ von Neumann algebra acting on a Hilbert space $\mathcal{H}$. We write type I_n algebra with $n \in \mathbb{N} \cup \{\infty\}$ for short. Furthermore, we assume that $\mathcal{M}$ has a cyclic and separating vector $u_0 \in \mathcal{H}$ and a (fixed) trace tr.

We cite the following results for convenience and to fix the notation. Proofs can be found e. g. in [KR86, § 9.3, § 9.4]:

Theorem 4.2.1. *Let $\mathcal{M}$ be a type I_n von Neumann algebra ($n \in \mathbb{N} \cup \{\infty\}$) acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector. Then $\mathcal{M}$ is unitarily equivalent to the algebra*

$$L(\mathcal{H}_0) \otimes \mathcal{A} \otimes \mathrm{I}_{\mathcal{H}_0}$$

acting on $\mathcal{H}_0 \otimes \mathcal{K} \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is an n-dimensional Hilbert space, and $\mathcal{A}$ is a maximal abelian algebra isomorphic to the center of $\mathcal{M}$ acting on the Hilbert space $\mathcal{K}$. Furthermore, the commutant of M is unitarily equivalent to

$$\mathrm{I}_{\mathcal{H}_0} \otimes \mathcal{A} \otimes L(\mathcal{H}_0).$$

Theorem 4.2.2. *Let $\mathcal{A}_c = L_\infty([0,1), \lambda)$ be the maximal abelian algebra of bounded functions on the unit interval with Lebesgue measure λ, and let $\mathcal{A}_j = L_\infty(S_j, \mu_j)$ be the maximal abelian algebra of bounded functions on a set S_j with j points and the counting measure μ_j ($j \in \mathbb{N} \cup \{\infty\}$), each acting on the corresponding Hilbert spaces $L_2(S, \mu)$ ($S \in \{[0,1), S_j\}$, $\mu \in \{\lambda, \mu_j\}$).*

Let $\mathcal{A}$ be a maximal abelian algebra. Then $\mathcal{A}$ is (unitarily equivalent to) one of the following algebras: $\mathcal{A}_c$, $\mathcal{A}_j$, or $\mathcal{A}_c \oplus \mathcal{A}_j$ ($j \in \mathbb{N} \cup \{\infty\}$).

Remark 4.2.3. Without further notice we often understand operators affiliated with an abelian algebra as measurable functions on an adequate measure space (S, μ). This is legitimated by Theorem 4.2.2. Furthermore, ω_μ denotes the n. s. f. weight given by the integral with respect to μ:

$$\omega_\mu(f) := \int_S f d\mu \quad (0 \le f \in L_\infty(S, \mu)). \tag{4.2.1}$$

Without loss of generality, we now consider only the type I_n von Neumann algebra $\mathcal{M} = L(\mathcal{H}_0) \otimes \mathcal{A} \otimes \mathrm{I}_{\mathcal{H}_0}$ acting on $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{K} \otimes \mathcal{H}_0$ with the notations of Theorem 4.2.1. Let Tr be the usual trace on $\mathcal{H}_0$ and ω_μ the n. s. f. weight defined by (4.2.1). Then $\mathrm{tr}_{\mathcal{M}}(\mathrm{M} \otimes \mathrm{A} \otimes \mathrm{I}) := \mathrm{Tr}(\mathrm{M})\mathrm{A}$ for $\mathrm{M} \in (L(\mathcal{H}_0))^+$ and $\mathrm{A} \in \mathcal{A}^+$ is the unique canonical central trace on $\mathcal{M}$ such that $\mathrm{tr}_{\mathcal{M}}(\mathrm{E}) = \mathrm{I}$ for all abelian projections $\mathrm{E} \in \mathcal{M}$ with central carrier I (cf. the remark following Theorem 2.3.10).

In Theorem 3.3.9 we proved that a modular operator Δ_0 corresponding to a cyclic and separating vector u_0 is the product of a positive invertible operator $\mathrm{H}_0 \eta \mathcal{M}$ with $\mathrm{tr}(\mathrm{H}_0) < \infty$ and $\mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0 \eta \mathcal{M}'$ (see Theorem 3.3.9). We therefore have to investigate the structure of such (finite trace) operators H_0. Since the trace tr was arbitrary we can choose

$$\mathrm{tr} = \omega_\mu \circ \mathrm{tr}_{\mathcal{M}}\,.$$

Note that for finite type I algebras with infinite-dimensional discrete center this is no finite trace, in contrast to the considerations of §3.2 where we assumed tr to be finite. Nevertheless, the proceeding of §3.3 can also applied to this case.

Theorem 4.2.4. *Let $\mathcal{M}$ be a type I_n von Neumann algebra ($n \in \mathbb{N} \cup \{\infty\}$) with trace $\mathrm{tr} = \omega_\mu \circ \mathrm{tr}_{\mathcal{M}}$. Assume $\mathrm{H}_0 \eta \mathcal{M}$ to be a positive invertible operator such that $\mathrm{tr}(\mathrm{H}_0) < \infty$. Then H_0 can be decomposed into*

$$\mathrm{H}_0 = \sum_{k \in K} f_k \mathrm{E}_k, \tag{4.2.2}$$

where $(\mathrm{E}_k)_{k \in K}$ is a family of pairwise orthogonal projections in $\mathcal{M}$ with sum I and $(f_k)_{k \in K}$ is a family of positive measurable functions affiliated with $\mathcal{Z}(\mathcal{M})$ ($K \subset \mathbb{N}$ is an appropriate index set). Moreover, $m_k := \mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k) \eta \mathcal{Z}(\mathcal{M})$ is an almost everywhere integer valued measurable function and the functions f_k have central carrier $\mathrm{C}_{f_k} = \mathrm{supp}(f_k) = \mathrm{C}_{\mathrm{E}_k}$ ($k \in K$). Furthermore,

$$\mathrm{tr}(\mathrm{H}_0) = \sum_{k \in K} \omega_\mu(f_k m_k) < \infty.$$

If we require additionally that $\mathrm{C}_{f_k} \geq \mathrm{C}_{f_j}$ and $f_k > f_j$ almost everywhere on the intersection of their supports for $k < j$ this decomposition is unique.

Remark 4.2.5. The statement of Theorem 4.2.4 becomes more evident if we consider the factor case. Then $\mathcal{M} \simeq L(\mathcal{H}_0)$, H_0 is a trace class operator on $\mathcal{H}_0$, the functions f_k are positive reals and the projections E_k are finite dimensional projections in $\mathcal{H}_0$ such that $m_k = \mathrm{Tr}(\mathrm{E}_k) = \dim(\mathrm{E}_k)$ and $\sum_k f_k m_k < \infty$. In the remainder of this section it is always instructive to have a look at the factor case in which many of the results are elementary.

Proof of Theorem 4.2.4. 1. We first assume $\mathcal{M}$ to be a factor. Theorem 4.2.1 then implies $\mathrm{H}_0 = \mathrm{H} \otimes \mathrm{I}$ where $0 < \mathrm{H} \eta L(\mathcal{H}_0)$ and $\mathrm{Tr}(\mathrm{H}) < \infty$. Hence, H is a trace class operator and has spectral decomposition

$$\mathrm{H} = \sum_{k \in K} f_k \mathrm{E}_k$$

where $(\mathrm{E}_k)_{k \in K}$ is a family of finite dimensional, pairwise orthogonal projections in $L(\mathcal{H}_0)$ and $(f_k)_{k \in K}$ is a family of non-zero positive real numbers. Furthermore, setting $m_k := \mathrm{Tr}(\mathrm{E}_k)$ we get $\sum_k m_k f_k < \infty$ since m_k is the dimension of E_k (it is therefore integer valued). If we require $f_k > f_l$ for $k < l$ this decomposition is obviously unique.

2. Let now $\mathcal{M}$ be an arbitrary type I_n algebra ($n \in \mathbb{N} \cup \{\infty\}$).

   (a) We decompose $\mathcal{H}$ as a direct integral with respect to the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ (see [KR86, Theorem 14.2.1]). The corresponding decomposition of $\mathcal{M}$ is a direct integral of type I_n factors $\mathcal{M}_p$ (see [KR86, Corollary 14.2.3]):

$$\mathcal{M} = \int_S \oplus \mathcal{M}_p d\mu(p)$$

(Note that $\mathcal{Z}(\mathcal{M}) \simeq L_\infty(S,\mu)$, see Theorem 4.2.2). Furthermore,

$$\mathrm{H}_0 = \int_S \oplus \mathrm{H}_p d\mu(p)$$

where $0 < \mathrm{H}_p \eta \mathcal{M}_p$ almost everywhere and

$$\mathrm{tr}_{\mathcal{M}} = \mathrm{Tr} \otimes \mathrm{id} = \int_S \oplus \, \mathrm{tr}_p \, \mathrm{I}_p d\mu(p)$$

where tr_p are traces on the factors $\mathcal{M}_p$ and I_p is the identity in $\mathcal{M}_p$. Since

$$\infty > \mathrm{tr}(\mathrm{H}_0) = \omega_\mu \circ \mathrm{tr}_{\mathcal{M}} = \int_S \mathrm{tr}_p(\mathrm{H}_p) d\mu(p)$$

$\mathrm{tr}_p(\mathrm{H}_p) < \infty$ for almost every $p \in S$. According to 1 almost every H_p can be decomposed uniquely into

$$\mathrm{H}_p = \sum_{k \in K_p} f_k^{(p)} \mathrm{E}_k^{(p)}.$$

Now define projections

$$\mathrm{E}_k := \int_S \oplus \mathrm{E}_k^{(p)} d\mu_p \in \mathcal{M}$$

for $k \in K = \bigcup_p K_p$ and functions

$$f_k(p) := f_k^{(p)} \eta \mathcal{Z}(\mathcal{M})$$

where $\mathrm{E}_k^{(p)} = 0$ and $f_k(p) = 0$ if $k \notin K_p$. Then

$$\mathrm{H}_0 = \sum_k f_k \mathrm{E}_k$$

is the claimed decomposition such that $\mathrm{C}_{f_k} = \mathrm{supp}(f_k) = \mathrm{C}_{\mathrm{E}_k}$ for $k \in K$, $f_k > f_l$ and $\mathrm{C}_{f_k} \geq \mathrm{C}_{f_j}$ and for $k < l$.

(b) We now prove that this decomposition is unique. For this purpose suppose that $\mathrm{H}_0 = \sum_{k \in K} f_k \mathrm{E}_k = \sum_{l \in L} g_l \mathrm{F}_l$ are two such decompositions. This implies

$$f_k \mathrm{E}_k \mathrm{F}_l = g_l \mathrm{E}_k \mathrm{F}_l \quad \text{for all } k \in K, l \in L$$

which is equivalent to $\mathrm{C}_{f_k - g_l} \mathrm{C}_{\mathrm{E}_k \mathrm{F}_l} = 0$ (cf. Theorem 4.1.2). Fix a $k_0 \in K$. Assume that a subprojection $\mathrm{P} \in \mathcal{Z}(\mathcal{M})$ of $\mathrm{C}_{\mathrm{E}_{k_0}}$ exists such that $\mathrm{P}\mathrm{C}_{f_{k_0} - g_l} = \mathrm{P}$ for all $l \in L$. Then $0 = \mathrm{P}\mathrm{C}_{f_{k_0} - g_l} \mathrm{C}_{\mathrm{E}_{k_0} \mathrm{F}_l} = \mathrm{P}\mathrm{C}_{\mathrm{E}_{k_0} \mathrm{F}_l}$ for all $l \in L$ and therefore

$$0 = \bigvee_{l \in L} \mathrm{P}\mathrm{C}_{\mathrm{E}_{k_0} \mathrm{F}_l} = \mathrm{P}\mathrm{C}_{\mathrm{E}_{k_0}} = \mathrm{P}$$

since $\sum_{l\in L} \mathrm{F}_l = \mathrm{I}$. It follows that for every non-zero central subprojection P of $\mathrm{C}_{\mathrm{E}_{k_0}}$ there is at least one l such that $\mathrm{PC}_{f_{k_0}-g_l} < \mathrm{P}$. Set now $k_0 = 1$ and $\mathrm{P} = \mathrm{C}_{\mathrm{E}_1}$. Then there is an index $l \in L$ such that

$$\mathrm{Q}_l := \mathrm{C}_{\mathrm{E}_1} - \mathrm{C}_{\mathrm{E}_1}\mathrm{C}_{f_1-g_l} > 0 \tag{4.2.3}$$

and $\mathrm{Q}_l\mathrm{C}_{f_1-g_l} = 0$. Interchanging the roles of the two compositions in the above considerations we obtain an index $k \in K$ and a non-zero central subprojection $\mathrm{R} \le \mathrm{Q}_l\mathrm{C}_{\mathrm{F}_1}$ such that $\mathrm{RC}_{f_k-g_1} = 0$. Now

$$\mathrm{R}g_1 = \mathrm{R}f_k \le \mathrm{R}f_1 = \mathrm{R}g_l \le \mathrm{R}g_1.$$

Hence, equality holds and thus $l = 1 = k$.
If $\mathrm{R}_0 := \mathrm{C}_{\mathrm{E}_1}\mathrm{C}_{f_1-g_1} \neq 0$ we get similarly central projections $\mathrm{S} \le \mathrm{R}_0$, $\mathrm{T} \le \mathrm{SC}_{\mathrm{F}_1}$ and indices k, l such that $\mathrm{SC}_{f_1-g_l} = 0$ and $\mathrm{TC}_{f_k-g_1} = 0$. Repeating the above argument we again get $l = 1 = k$ which is a contradiction. Hence $\mathrm{R}_0 = 0$ which implies $f_1 = g_1$ and $\mathrm{E}_1 = \mathrm{F}_1$.
Repeating this argument for the other indices $k \in K$, we get the uniqueness.

(c) Furthermore,

$$\begin{aligned}\infty > \mathrm{tr}(\mathrm{H}_0) &= \omega_\mu(\mathrm{tr}_{\mathcal{M}}(\mathrm{H}_0))\\ &= \omega_\mu(\mathrm{tr}_{\mathcal{M}}(\sum_k f_k\mathrm{E}_k))\\ &= \sum_k \omega_\mu(f_k\,\mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k))\\ &= \sum_k \omega_\mu(f_k m_k)\end{aligned}$$

and

$$m_k(p) = \mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k)(p) = \mathrm{tr}_p(\mathrm{E}_k^{(p)})$$

is a positive almost everywhere integer valued measurable function.

□

Defining

$$f_k^{-1}(p) := \begin{cases} 1/f_k(p) & \text{if } f_k(p) \neq 0 \\ 0 & \text{if } f_k(p) = 0 \end{cases}, \tag{4.2.4}$$

we get functions $f_k^{-1}\eta\mathcal{Z}(\mathcal{M})$ such that $\mathrm{H}_0^{-1} = \sum_{k\in K} f_k^{-1}\mathrm{E}_k$. Note that $\mathrm{supp}(f_k) = \mathrm{C}_{\mathrm{E}_k}$. Therefore, it is immaterial how we define f_k^{-1} on $S \setminus \mathrm{supp}(f_k)$. Theorem 4.2.4 and (4.2.4) now yield the following decomposition for Δ_0:

$$\begin{aligned}\Delta_0 &= \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\\ &= \sum_{k,l\in K} (f_k\mathrm{E}_k)\mathrm{J}_0(f_l^{-1}\mathrm{E}_l)\mathrm{J}_0\\ &= \sum_{k,l\in K} f_k f_l^{-1}(\mathrm{E}_k\mathrm{J}_0\mathrm{E}_l\mathrm{J}_0)\\ &=: \sum_{j\in J} g_j\mathrm{F}_j\end{aligned} \tag{4.2.5}$$

where every function g_j equals at least one of the products $f_k f_l^{-1}$, $g_j \neq g_i$ for $i \neq j$, and the projections F_j are pairwise orthogonal.

Lemma 4.2.6. *With the notations introduced above, Δ_0 has the spectrum*

$$\sigma(\Delta_0) = \overline{\bigcup_{j \in J} \sigma(g_j)} = \overline{\bigcup_{k,l \in K} \sigma(f_k f_l^{-1})}, \tag{4.2.6}$$

where the spectrum $\sigma(f)$ of the measurable function f is the closure of the essential range of f.

Remark 4.2.7. As in Theorem 4.2.4 the factor case is elementary. We then have $\sigma(f_k) = f_k \in \mathbb{R}$ and Lemma 4.2.6 is equivalent to Corollary 4.1.3.

Proof of Lemma 4.2.6. If the intersection of the supports of f_k and f_l $(k, l \in K)$ is non-null set we get

$$\mathrm{C}_{\mathrm{E}_k} \mathrm{C}_{\mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0} = \mathrm{C}_{\mathrm{E}_k} \mathrm{C}_{\mathrm{E}_l} = \mathrm{supp}(f_k f_l) \neq 0.$$

This implies $\mathrm{E}_k \mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0 \neq 0$ according to Theorem 4.1.2. The assertion now follows directly from (4.2.5) and

$$\mathrm{F}_j = \sum_{\{f_k f_l^{-1} = g_j\}} \mathrm{E}_k \mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0 \neq 0,$$

where the summation extends over all indices k, l such that $f_k f_l^{-1} = g_j$, i.e. $\{f_k f_l^{-1} = g_j\} := \{k, l \in K | f_k f_l^{-1} = g_j\}$. □

In the next proposition we collect some further properties of the projections F_j appearing in (4.2.5).

Proposition 4.2.8. *Each projection F_j commutes with $\mathcal{B} := \mathrm{I}_{\mathcal{H}_0} \otimes \mathcal{A} \otimes \mathrm{I}_{\mathcal{H}_0}$ and has central carrier in $\mathcal{B}$ equal to the support of the corresponding g_j.*

Proof. The first assertion follows from

$$\mathrm{F}_j = \sum_{\{f_k f_l^{-1} = g_j\}} \mathrm{E}_k \mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0$$

since $\mathrm{E}_k \in \mathcal{M} \subset \mathcal{B}'$ and $\mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0 \in \mathcal{M}' \subset \mathcal{B}'$.

To prove the second assertion, we first note that the central carrier $\mathrm{C}_{\mathrm{F}_j}$ of F_j is

$$\mathrm{C}_{\mathrm{F}_j} = \bigvee_{\{f_k f_l^{-1} = g_j\}} \mathrm{C}_{\mathrm{E}_k \mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0}.$$

We now prove

$$\mathrm{C}_{\mathrm{E}_k \mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0} = \mathrm{C}_{\mathrm{E}_k} \mathrm{C}_{\mathrm{E}_l}.$$

On the contrary, we suppose that there is projection $\mathrm{Q} \in \mathcal{P}(\mathcal{B})$ such that $\mathrm{C}_{\mathrm{E}_k} \mathrm{C}_{\mathrm{E}_l} > \mathrm{Q}$ and

$$\mathrm{E}_k \mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0 \mathrm{Q} = \mathrm{E}_k \mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0.$$

Then

$$\mathrm{E}_k(\mathrm{I}-\mathrm{Q})\mathrm{J}_0\mathrm{E}_l\mathrm{J}_0 = 0$$

which is equivalent to

$$\mathrm{C}_{\mathrm{E}_k(\mathrm{I}-\mathrm{Q})}\mathrm{C}_{\mathrm{J}_0\mathrm{E}_l\mathrm{J}_0} = (\mathrm{I}-\mathrm{Q})\mathrm{C}_{\mathrm{E}_k}\mathrm{C}_{\mathrm{E}_l} = 0.$$

This implies $\mathrm{Q} \geq \mathrm{C}_{\mathrm{E}_k}\mathrm{C}_{\mathrm{E}_l}$, a contradiction. We have hence proved that

$$\mathrm{C}_{\mathrm{F}_j} = \bigvee_{\{f_k f_l^{-1} = g_j\}} \mathrm{C}_{\mathrm{E}_k}\mathrm{C}_{\mathrm{E}_l}.$$

On the other hand, the support of g_j is

$$\mathrm{supp}(g_j) = \bigvee_{\{f_k f_l^{-1} = g_j\}} \mathrm{supp}(f_k f_l^{-1}) = \bigvee_{\{f_k f_l^{-1} = g_j\}} \mathrm{C}_{\mathrm{E}_k}\mathrm{C}_{\mathrm{E}_l},$$

as well. □

Set now $n_j := \mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j)$ $(j \in J)$ where

$$\mathrm{tr}_{\mathcal{B}'} = \mathrm{Tr}\otimes\mathrm{id}\otimes\mathrm{Tr} = (\mathrm{Tr}\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{Tr}) = \mathrm{tr}_{\mathcal{M}}\,\mathrm{tr}_{\mathcal{M}'}$$

is a center valued trace on the commutant $\mathcal{B}' = L(\mathcal{H}_0)\otimes\mathcal{A}\otimes L(\mathcal{H}_0)$ of $\mathcal{B}$. We call m_k defined in Theorem 4.2.4 the *central multiplicity of* E_k *in* $\mathcal{M}$ and n_j the *central multiplicity of* F_j *in* $\mathcal{B}'$. In the factor case these multiplicities are the usual multiplicities of eigenvalues of selfadjoint operators on the Hilbert spaces $\mathcal{H}_0$ (for H_0) and $\mathcal{H}$ (for Δ), respectively, which are the dimensions of the corresponding eigenspaces.

Proposition 4.2.9. *Let $(n_j)_{j\in J}$, $(g_j)_{j\in J}$, $(m_k)_{k\in K}$, and $(f_k)_{k\in K}$ be defined as above. Then*

$$n_j = \sum_{\{f_k f_l^{-1} = g_j\}} m_k m_l \quad \text{for all } j \in J. \tag{4.2.7}$$

Proof. Note first that

$$\begin{aligned} n_j = \mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j) &= \sum_{\{f_k f_l^{-1} = g_j\}} \mathrm{tr}_{\mathcal{B}'}(\mathrm{E}_k\mathrm{J}_0\mathrm{E}_l\mathrm{J}_0) \\ &= \sum_{\{f_k f_l^{-1} = g_j\}} \mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k)\,\mathrm{tr}_{\mathcal{M}'}(\mathrm{J}_0\mathrm{E}_l\mathrm{J}_0). \end{aligned}$$

Since $\mathrm{tr}_{\mathcal{M}'}(\mathrm{J}_0(\cdot)\mathrm{J}_0)$ is a central trace on $\mathcal{M}$ there is a positive invertible $\mathrm{C}\eta\mathcal{B}$ such that

$$\mathrm{tr}_{\mathcal{M}} = \mathrm{C}\,\mathrm{tr}_{\mathcal{M}'}(\mathrm{J}_0(\cdot)\mathrm{J}_0)$$

(see Remark 2.3.11). Furthermore, let $\mathrm{F} := \mathrm{E}\otimes\mathrm{I}_{\mathcal{K}}\otimes\mathrm{I}_{\mathcal{H}_0} \in \mathcal{M}$ where $\mathrm{E} \in L(\mathcal{H}_0)$ is a one-dimensional projection. $\mathrm{J}_0\mathrm{F}\mathrm{J}_0$ and $\mathrm{I}\otimes\mathrm{I}\otimes\mathrm{E}$ are both abelian projections in $\mathcal{M}'$ with central carrier I. Since all abelian projections with the same central carrier are equivalent (see [KR86, Proposition 6.4.6]) we get (cf. (2.3.2))

$$\mathrm{I} = \mathrm{tr}_{\mathcal{M}}(\mathrm{F}) = \mathrm{C}\,\mathrm{tr}_{\mathcal{M}'}(\mathrm{J}_0\mathrm{F}\mathrm{J}_0) = \mathrm{C}\,\mathrm{tr}_{\mathcal{M}'}(\mathrm{I}_{\mathcal{H}_0}\otimes\mathrm{I}_{\mathcal{K}}\otimes\mathrm{E}) = \mathrm{C}.$$

Hence,

$$\mathrm{tr}_{\mathcal{M}'}(\mathrm{J}_0\mathrm{E}_l\mathrm{J}_0) = \mathrm{tr}_{\mathcal{M}}(\mathrm{E}_l).$$

(4.2.7) now follows immediately. □

(4.2.5), Proposition 4.2.9, and Proposition 4.2.8 legitimate the following

Definition 4.2.10. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B} \subset L(\mathcal{H})$ be an abelian algebra with commutant of type I_{n^2} ($n \in \mathbb{N} \cup \{\infty\}$).

1. A positive invertible operator Δ acting on $\mathcal{H}$ is called *n-decomposable with respect to $\mathcal{B}$* if there exist a family $(\mathrm{F}_j)_{j\in J}$ of pairwise orthogonal projections in $\mathcal{B}'$ and a family $(g_j)_{j\in J}$ of positive elements affiliated with $\mathcal{B}$ ($J \subset \mathbb{N}$ an index set) fulfilling the following properties:
   - $\sum_j \mathrm{F}_j = \mathrm{I}$,
   - $\Delta = \sum_j g_j \mathrm{F}_j$,
   - the central carrier of F_j with respect to $\mathcal{B}'$ coincides with the central carrier of g_j for all $j \in J$.
2. Let $\Delta = \sum_j g_j \mathrm{F}_j$ be an operator which is n-decomposable with respect to $\mathcal{B}$. Set $n_j := \mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j)$ where $\mathrm{tr}_{\mathcal{B}'}$ is the canonical central trace on $\mathcal{B}'$. Then Δ possesses *multiplicative central spectrum of type I_n* if there are families $(f_k)_{k\in K}$ and $(m_k)_{k\in K}$ of positive elements affiliated with $\mathcal{B}$ fulfilling the following properties:
   - m_k is (as a function) integer valued almost everywhere,
   - $\sum_{k\in K} m_k = n\mathrm{I}$,
   - $\sum_{k\in K} \omega_\mu(m_k f_k) < \infty$,
   - for every $j \in J$ there exists at least one pair $(k,l) \in K^2$ such that $g_j = f_k f_l^{-1}$ and
   $$n_j = \sum_{\{f_k f_l^{-1} = g_j\}} m_k m_l \quad \text{for all } j \in J.$$

Remark 4.2.11. 1. In the factor case, Definition 4.2.10 means the following. Let Δ be a positive invertible operator acting on an n^2-dimensional Hilbert space $\mathcal{H}$. It is n-decomposable with respect to $\mathbb{C}$ if and only if it possesses *pure point spectrum*, i. e. if and only if the spectrum is the closure of the set of all eigenvalues and $\mathcal{H}$ is spanned by the eigenprojections. With the notations of Definition 4.2.10 we have $\mathcal{B} = \mathbb{C}$, $g_j \in \mathbb{R}_{>0}$, and $\Delta = \sum_j g_j \mathrm{F}_j$ is the spectral decomposition of Δ.

Moreover, Δ possesses multiplicative central spectrum if and only if a l_1-family $(f_k)_{k\in K}$ in $\mathbb{R}_{>0}$ exists such that the eigenvalues g_j of Δ fulfil

$$\{g_j | j \in J\} = \{f_k f_l^{-1} | k, l \in K\},$$

where the eigenvalues are repeated according to their multiplicities.

2. The considerations preceding Definition 4.2.10 show that the modular operators for type I_n algebras corresponding to cyclic and separating vectors are n-decomposable operators with multiplicative spectrum of type I_n.

Theorem 4.2.12. *Let Δ be a positive invertible operator acting on a Hilbert space $\mathcal{H}$. Δ is a modular operator for a von Neumann algebra of type I_n, $n \in \mathbb{N} \cup \{\infty\}$, with center (isomorphic to) $\mathcal{B}$ corresponding to a cyclic and separating vector if and only if Δ is an n-decomposable operator with respect to $\mathcal{B}$ which possesses multiplicative central spectrum of type I_n.*

Proof. If Δ is a modular operator then Δ is an n-decomposable operator with respect to $\mathcal{B}$ which possesses multiplicative central spectrum of type I_n (see Remark 4.2.11.2).

We now prove the converse. Since $\mathcal{B}$ is an abelian algebra with commutant of type I_{n^2} it is unitarily equivalent to $\mathcal{A} \otimes \mathrm{I}_{\mathcal{H}_1}$ acting on $\mathcal{K} \otimes \mathcal{H}_1$ where $\mathcal{H}_1$ is an n^2-dimensional Hilbert space and $\mathcal{A}$ is a maximal abelian algebra isomorphic to $\mathcal{B}$ which acts on $\mathcal{K}$ (see Theorem 4.2.1). $\mathcal{H}_1$ can be decomposed into a tensor product of Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_0 \otimes \mathcal{H}_0$ where $\mathcal{H}_0$ is an n-dimensional Hilbert space.

We consider the von Neumann algebra $\mathcal{M} := \mathcal{A} \otimes L(\mathcal{H}_0) \otimes \mathrm{I}_{\mathcal{H}_0}$ which acts on $\mathcal{K} \otimes \mathcal{H}_0 \otimes \mathcal{H}_0$. $\mathcal{M}$ is a type I_n algebra with center isomorphic to $\mathcal{A}$. Moreover, $\mathcal{M} \subset \mathcal{B}' = \mathcal{A} \otimes L(\mathcal{H}_0) \otimes L(\mathcal{H}_0)$ and $\mathcal{M}$ has the canonical central trace $\mathrm{tr}_{\mathcal{M}} := \mathrm{id} \otimes \mathrm{Tr} \otimes \mathrm{id}$.

Consider now $\mathcal{M} = \mathcal{A} \otimes L(\mathcal{H}_0)$ as $L_\infty(S, \mu; L(\mathcal{H}_0))$. Since m_k is an almost everywhere integer-valued function, there are projections $\mathrm{E}_k(p) \in L(\mathcal{H}_0)$ such that $\dim(\mathrm{E}_k(p)) = m_k(p)$ almost everywhere. Defining $\mathrm{E}_k = \int_S \oplus \mathrm{E}_k^{(p)} d\mu(p)$ we obtain a family $(\mathrm{E}_k)_{k \in K}$ of pairwise orthogonal projections in $\mathcal{M}$ such that $\mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k) = m_k$ $(k \in K)$. Setting

$$\mathrm{H}_0 := \sum_{k \in K} f_k \mathrm{E}_k,$$

we get a positive invertible operator affiliated with $\mathcal{M}$ such that

$$\begin{aligned} \mathrm{tr}(\mathrm{H}_0) &= \omega_\mu(\mathrm{tr}_{\mathcal{M}}(\mathrm{H}_0)) \\ &= \sum_{k \in K} \omega_\mu(f_k \, \mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k)) \\ &= \sum_{k \in K} \omega_\mu(f_k m_k) < \infty. \end{aligned}$$

According to §3.2, if n is finite, or §3.3, if n is infinite, there is now a cyclic and separating vector u_0 (the vector corresponding to $\mathrm{H}_0^{1/2}$ in the GNS representation with respect to the trace) for $\mathcal{M}$ such that $\Delta_{u_0} = \mathrm{H}_0 \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0$ is the modular operator corresponding to u_0 with an appropriate modular conjugation J_0. The same calculations as in (4.2.5) and in the proof of Proposition 4.2.9 yield $\Delta_{u_0} = \sum_{j \in J} g_j \hat{\mathrm{F}}_j$, where $\hat{\mathrm{F}}_j \in \mathcal{B}'$ with $\mathrm{tr}_{\mathcal{B}'}(\hat{\mathrm{F}}_j) = n_j$ $(j \in J)$. Since also $\mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j) = n_j = \mathrm{tr}_{\mathcal{B}'}(\hat{\mathrm{F}}_j)$, we infer that $\hat{\mathrm{F}}_j \sim \mathrm{F}_j$ in $\mathcal{B}'$ (cf. (2.3.2)), i.e. there are partial isometries V_j such that $\mathrm{V}_j\mathrm{V}_j^* = \hat{\mathrm{F}}_j$ and $\mathrm{V}_j^*\mathrm{V}_j = \mathrm{F}_j$. Setting $\mathrm{U} := \sum \mathrm{V}_j$, we get a unitary such that

$$\mathrm{U}\Delta\mathrm{U}^* = \sum_j \mathrm{V}_j g_j \mathrm{F}_j \mathrm{V_j}^* = \sum_j g_j \hat{\mathrm{F}}_j = \Delta_{u_0}.$$

Hence, U^*u_0 is a cyclic and separating vector for $\mathrm{U}^*\mathcal{M}\mathrm{U}$ with modular operator $\mathrm{U}^*\Delta_{u_0}\mathrm{U} = \Delta$ (see Corollary 2.1.5). □

The proof of Theorem 4.2.12 also implies the following

Corollary 4.2.13. *Let Δ be a modular operator for a type I_n algebra $\mathcal{M}$ corresponding to a cyclic and separating vector u_0. Suppose that $(m_k)_{k\in K}$ and $(f_k)_{k\in K}$ are two sequences fulfilling the prerequisites of Definition 4.2.10.2 with respect to Δ. Then there is a cyclic and separating vector u for $\mathcal{M}_0$ such that the corresponding modular operator Δ_u is unitarily equivalent to Δ and $\Delta_u = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$. Moreover, $\mathrm{H} = \sum_k f_k\mathrm{E}_k$ is the decomposition of $\mathrm{H}\eta\mathcal{M}$ where $m_k = \mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k)$.*

4.3 Type II Algebras

The spectral theory of modular operators is much more involved in the type II case than in the type I case, due to the fact that we have no longer the tensor product decomposition (4.2.5) of finite trace operators. Nevertheless, some results can also be extended to the type II case if we restrict ourselves to the case of diagonalizable cyclic and separating vectors (see the definition below).

In this section $\mathcal{M}$ is always a (finite or properly infinite) type II algebra (an algebra of type II_1 or type II_∞ or, for short, of type II_n $(n \in \{1,\infty\})$) which acts on a Hilbert space $\mathcal{H}$ and has cyclic and separating vector $u_0 \in \mathcal{H}$ and central trace $\mathrm{tr}_{\mathcal{M}}$. Furthermore, the center of $\mathcal{M}$ is isomorphic to $L_\infty(S,\mu)$ with an appropriate measure space (S,μ) (see Theorem 4.2.2). For simplicity of notation, we will often use the same letter f for measurable functions on S as well as for the corresponding elements affiliated with $\mathcal{Z}(\mathcal{M})$. We choose $\mathrm{tr} = \omega_\mu \circ \mathrm{tr}_{\mathcal{M}}$ as the arbitrary tracial weight tr on $\mathcal{M}$ (cf. the remark preceding Theorem 4.2.4).

Whereas we could prove in the type I case that every operator with finite trace can be decomposed in a sum (see (4.2.2)), this decomposition is in general not possible in the type II case. Therefore, we will first consider the subclass of operators (and corresponding cyclic and separating vectors) which allow such a decomposition. We will treat the spectral theory of this case in detail in the first subsection. In the second subsection we will make some remarks on the general case.

4.3.1 The Diagonalizable Case

We first introduce the notion of operators diagonalizable with respect to the center:

Definition 4.3.1. Let $\mathrm{H}_0\eta\mathcal{M}$ be a positive invertible operator.

1. We call H_0 *diagonalizable with respect to the center of* $\mathcal{M}$ if there are a family $(f_k)_{k\in K}$ of positive elements affiliated with $\mathcal{Z}(\mathcal{M})$ and a family $(\mathrm{E}_k)_{k\in K}$ of pairwise orthogonal projections in $\mathcal{M}$ such that
   - $\sum_{k\in K}\mathrm{E}_k = \mathrm{I}$,
   - $\mathrm{C}_{\mathrm{E}_k} = \mathrm{C}_{f_k}$ $(k \in K)$,
   - and $\mathrm{H}_0 = \sum_{k\in K} f_k\mathrm{E}_k$.

2. Let $u_0 \in \mathcal{H}$ be a cyclic and separating vector for $\mathcal{M}$ such that the modular operator Δ_0 corresponding to u_0 is $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$ (J_0 is the modular conjugation corresponding to u_0, cf. Theorem 3.3.9). u_0 is *diagonalizable with respect to the center of* $\mathcal{M}$ if H_0 is diagonalizable with respect to the center of $\mathcal{M}$.

Remark 4.3.2.

1. If we require for $k < j$ that $\mathrm{C}_{f_k} \geq \mathrm{C}_{f_j}$ and $f_k > f_j$ almost everywhere on the intersection of their supports as in Theorem 4.2.4, then the decomposition $\sum_k f_k\mathrm{E}_k$ is unique. The proof is the same as in the type I case.

2. As in §4.2 we treat the factor case separately. Definition 4.3.1 then means that the operator H_0 possesses pure point spectrum in the sense of Remark 4.2.11, i.e. $f_k \in \mathbb{R}_{>0}$ for all $k \in K$, $\sigma(\mathrm{H}_0) = \overline{\{f_k | k \in K\}}$, and $\sum_{k\in K}\mathrm{E}_k = \mathrm{I}$. The remarks of §4.2 concerning the factor case now also apply to the results presented in this section.

The following example shows that we can not hope to obtain an analogue of Theorem 4.2.4 in the type II case.

Example 4.3.3. Let $\mathcal{R} = \mathcal{R}(\mathcal{A},\alpha)$ be the crossed product algebra of $\mathcal{A} = L_\infty(S,\lambda)$, where λ is the Lebesgue measure on the Borel-σ-algebra on $S = [0,1)$ and α is the representation of the group G of all rational translations, modulo 1, of S, which we considered in Example 2.2.6. $\mathcal{R}$ is a factor of type II_1. Let $\mathrm{M} = [\mathrm{U}(pq^{-1})\mathrm{A}(pq^{-1})]_{p,q\in G} \in \mathcal{R}$ be an arbitrary element in $\mathcal{R}$, where U is a unitary representation of G on $\mathcal{H}$ which implements α and $\mathrm{A}(p) \in \mathcal{A} = L_\infty(S,\lambda)$ for all $p \in G$. The unique trace tr on $\mathcal{R}$ is

$$\mathrm{tr}(\mathrm{M}) = \int \mathrm{A}(e)d\lambda.$$

Let Φ be the homomorphism from $\mathcal{A}$ into $\mathcal{R}$ defined by $\Phi(f) = [\delta_{p,q}\mathrm{M}_f]_{p,q}$ for $f \in \mathcal{A}$ (M_f is the multiplication operator corresponding to f, in the following we write f for brevity). Then, $\mathrm{tr}(\Phi(f)) < \infty$ if and only if $\int f d\lambda < \infty$ for $f \geq 0$. Let now $0 < f \in \mathcal{A}$ be such that f^{-1} exists and is bounded. Assume further that f is strictly monotone increasing and

$$\int_0^1 f(x)d\lambda(x) < \infty.$$

We can choose $f = \exp$, for instance. This implies $\Phi(f), \Phi(f)^{-1} \in \mathcal{R}$ and $\mathrm{tr}(\Phi(f)) < \infty$. Moreover, the spectral measure of $\Phi(f)$ is

$$\mathrm{E}_{\Phi(f)}(B) = [\chi_{f^{-1}(B)}\delta_{p,q}]_{p,q},$$

where χ_M is the characteristic function of the Borel set M. Hence, $\Phi(f)$ has the same spectrum as f and $\Phi(f)$ can not be decomposed into $\mathrm{H}_0 = \sum_{k\in K} f_k \mathrm{E}_k$ with $f_k \in \mathcal{Z}(\mathcal{R}) = \mathbb{C}$. The latter follows since f has no eigenvalues.

Similar counterexamples can also be constructed for the type II_∞ case (see Example 2.2.7). Counterexamples with non-trivial centers can be obtained if one considers the tensor product of factors with an abelian algebra.

From now on we make the assumption that the type II von Neumann algebra $\mathcal{M}$ has a cyclic and separating vector u_0 which is diagonalizable with respect to the center.

Definition 4.3.1 and (4.2.4) then yield the following decomposition for the modular operator Δ_0 corresponding to u_0 (cf. (4.2.5)):

$$\begin{aligned}\Delta_0 &= \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0 \\ &= \sum_{k,l\in K} (f_k\mathrm{E}_k)\mathrm{J}_0(f_l^{-1}\mathrm{E}_l)\mathrm{J}_0 \\ &= \sum_{k,l\in K} f_k f_l^{-1}(\mathrm{E}_k\mathrm{J}_0\mathrm{E}_l\mathrm{J}_0) \\ &=: \sum_{j\in J} g_j \mathrm{F}_j,\end{aligned} \tag{4.3.1}$$

where every function g_j equals at least one of the the products $f_k f_l^{-1}$, $g_j \neq g_i$ for $i \neq j$, and the projections F_j are pairwise orthogonal.

Lemma 4.3.4. *With the notations introduced above, Δ_0 has the spectrum*

$$\sigma(\Delta_0) = \overline{\bigcup_{j\in J}\sigma(g_j)} = \overline{\bigcup_{k,l\in K}\sigma(f_k f_l^{-1})}. \tag{4.3.2}$$

Proof. The proof is exactly the same as that of Lemma 4.2.6. □

The next proposition can also be proved in exactly the same way as the corresponding Proposition 4.2.8.

Proposition 4.3.5. *Each projection F_j commutes with $\mathcal{B} := \mathcal{Z}(\mathcal{M})$ and has central carrier in $\mathcal{B}$ equal to the support of the corresponding g_j.*

Let now $\mathrm{tr}_{\mathcal{B}'}$ be the central trace on the commutant $\mathcal{B}'$ of $\mathcal{B} = \mathcal{Z}(\mathcal{M})$ uniquely determined by the condition that $\mathrm{tr}_{\mathcal{B}'}(\mathrm{E}) = \mathrm{I}$ for all abelian projections $\mathrm{E} \in \mathcal{B}'$ with central carrier I. As in §4.2, we set $n_j := \mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j)$ $(j \in J)$ and call it the *central multiplicity of* F_j *in* $\mathcal{B}'$. The functions m_k defined in Definition 4.3.1 are the *central multiplicities of* E_k *in* $\mathcal{M}$.

Remark 4.3.6. In contrast to the type I case, the central multiplicities of the projections E_k in $\mathcal{M}$ are not the usual dimensions of the corresponding subspaces, also in the factor case. This is implied by the observation that the central multiplicities of a type II factor are positive real valued (see e. g. [KR86, 8.4.4]), whereas the dimension only takes values in the positive integers. Moreover, the dimensions of the subspaces corresponding to projections in type II factors are always infinite (cf. e. g. [KR86, Lemma 6.5.6]).

Proposition 4.3.7. *With the notations introduced above, we have*

$$n_j = \mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j) = \infty \cdot \mathrm{I}$$

for all $j \in J$.

Proof.

1. Consider first the factor case. Hence, $\mathcal{B} = \mathcal{Z}(\mathcal{M}) = \mathbb{C}$ and the numbers $g_j \in \mathbb{C}$ are the eigenvalues of Δ_0. Lemma 4.1.6 then implies that all eigenvalues of Δ_0 are of infinite (central) multiplicity. Hence, the corresponding projections F_j are the sum of infinite many minimal (equivalent) projections in $\mathcal{B}' = L(\mathcal{H})$, namely of infinite many one-dimensional projections. This implies $\mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j) = \infty$ for all $j \in J$.

2. In the non-factor case decompose $\mathcal{H} = \oplus \int \mathcal{H}_p d\mu(p)$ into a direct integral with respect to the algebra $\mathcal{B}$ (see [KR86, Theorem 14.2.1] and Theorem 4.2.1). $\mathcal{B}'$ is then the algebra of decomposable operators on $\mathcal{H}$, and $\mathcal{M}, \mathcal{M}' \subset \mathcal{B}'$. The decompositions

$$\int_S \oplus \mathcal{M}_p d\mu(p) \quad \text{and} \quad \int_S \oplus \mathcal{M}'_p d\mu(p)$$

of $\mathcal{M}$ and $\mathcal{M}'$ are direct integrals of type II_n factors ($n \in \{1, \infty\}$) (see [KR86, Corollary 14.2.3]). Then $\mathrm{E}_k =: \mathrm{P} \in \mathcal{M}$ and $\mathrm{J}_0\mathrm{E}_l\mathrm{J}_0 =: \mathrm{Q} \in \mathcal{M}'$ can be decomposed into

$$\mathrm{P} = \int_S \oplus \mathrm{P}_p d\mu(p) \quad \text{and} \quad \mathrm{Q} = \int_S \oplus \mathrm{Q}_p d\mu(p),$$

where $\mathrm{P}_p \in \mathcal{M}_p$, $\mathrm{Q}_p \in \mathcal{M}'_p$ are projections almost everywhere. The first paragraph of the proof implies that almost every $\mathrm{P}_p\mathrm{Q}_p$ is the sum of infinite many equivalent projections in $L(\mathcal{H}_p)$, say $\mathrm{P}_p\mathrm{Q}_p = \sum_{n\in\mathbb{N}} \mathrm{G}_p^n$. Defining $\mathrm{G}_n := \int_S \oplus \mathrm{G}_p^n d\mu(p) \in \mathcal{B}'$ for $n \in \mathbb{N}$, we get a countable infinite family of equivalent projections in $\mathcal{B}'$ such that $\mathrm{PQ} = \sum_{n\in\mathbb{N}} \mathrm{G}_n$. Hence

$$\mathrm{tr}_{\mathcal{B}'}(\mathrm{PQ}) = \sum_{n\in\mathbb{Z}} \mathrm{tr}_{\mathcal{B}'}(\mathrm{G}_n) = \infty \cdot \mathrm{tr}_{\mathcal{B}'}(\mathrm{G}_1) = \infty \cdot \mathrm{I}.$$

□

Remark 4.3.8.

1. Note the following difference to the type I case: In the type II case there is no relation between the multiplicities n_j of the modular operator $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$, which are infinite almost everywhere, and the multiplicities m_k of the operator $\mathrm{H}_0\eta\mathcal{M}$ (cf. (4.2.7)), which are finite almost everywhere.

2. Lemma 4.3.4, Proposition 4.3.5, and Proposition 4.3.7 imply that modular operators for type II algebras corresponding to cyclic and separating vectors, which are diagonalizable with respect to the center, are ∞-decomposable operators with respect to $\mathcal{B}$ in the sense of Definition 4.2.10.

The preceding considerations motivate the following analogue of Definition 4.2.10.

Definition 4.3.9. Let Δ be a positive invertible operator acting on a Hilbert space $\mathcal{H}$. Suppose that $\Delta = \sum_{j \in J} g_j \mathrm{F}_j$ is ∞-decomposable with respect to an abelian algebra $\mathcal{B}$.

1. We say Δ has *uniformly infinite central multiplicity* if each F_j has infinite central multiplicity, i. e. $\mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j) = \infty$ where $\mathrm{tr}_{\mathcal{B}'}$ is the canonical central trace of $\mathcal{B}'$.

2. Suppose that Δ has uniformly infinite central multiplicity. Then Δ possesses *multiplicative central spectrum of type II_n* $(n \in \{1, \infty\})$ if there are families $(f_k)_{k \in K}$ and $(m_k)_{k \in K}$ of positive elements affiliated with $\mathcal{B}$ fulfilling the following properties:

    - $\sum_{k \in K} m_k = n\mathrm{I}$,
    - $\sum_{k \in K} \omega_\mu(m_k f_k) < \infty$,
    - for every $j \in J$ there exists at least one pair $(k, l) \in K^2$ such that $g_j = f_k f_l^{-1}$.

Remark 4.3.10. 1. An operator Δ which is ∞-decomposable with respect to $\mathbb{C}$ is an operator with pure point spectrum on an infinite-dimensional Hilbert space according to Remark 4.2.11. Hence, Definition 4.3.9 means for $\mathcal{B} = \mathbb{C}$, which corresponds to the factor case, that every eigenvalue of Δ has infinite multiplicity (the corresponding eigenspace is infinite-dimensional). Furthermore, Δ possesses multiplicative central spectrum of type II_n if and only if there are families of positive non-zero reals $(f_k)_k$ and $(m_k)_k$ such that $\sum_{k \in K} m_k = n$, $\sum_{k \in K} m_k f_k < \infty$, and every eigenvalue of Δ is a product $f_k f_l^{-1}$ for some pair of indices (k, l).

2. According to the above considerations the modular operators for type II_n algebras corresponding to cyclic and separating vectors diagonalizable with respect to the center are ∞-decomposable operators with uniformly infinite central multiplicity and with multiplicative spectrum of type II_n.

Before we formulate and prove the converse of Remark 4.3.10.2 we show the following

Proposition 4.3.11. *Let $\mathcal{M}$ be a type II_n von Neumann factor, $n \in \{1, \infty\}$, with trace* tr. *For every countable family $(m_k)_{k \in K}$ of positive reals such that $\sum_k m_k = n$ there exists a family of pairwise orthogonal projections $(\mathrm{E}_k)_{k \in K}$ in $\mathcal{M}$ such that $\mathrm{tr}(\mathrm{E}_k) = m_k$ for every k and $\sum_k \mathrm{E}_k = \mathrm{I}$.*

Proof. We construct the family $(\mathrm{E}_k)_{k\in K}$ inductively. Since the range of $\mathrm{tr}_{|\mathcal{P}(\mathcal{M})}$ is $[0,1]$ if $\mathcal{M}$ is a type II_1 factor ($\mathbb{R}_{\geq 0}$ if $\mathcal{M}$ is a type II_∞ factor) (cf. e.g. [KR86, 8.4.4]) there is a projection E_1 in $\mathcal{M}$ such that $\mathrm{tr}(\mathrm{E}_1) = m_1$.

Suppose now that we have constructed a family $(\mathrm{E}_k)_{1\leq k<N}$ for $N \in K$ such that the projections E_k are pairwise orthogonal and $\mathrm{tr}(\mathrm{E}_k) = m_k$ $(1 \leq k < N)$. If we set $\mathrm{F}_N := \mathrm{Id} - \sum_{k=1}^N \mathrm{E}_k$ the restricted algebra $\mathrm{F}_N\mathcal{M}\mathrm{F}_N$ is again a type II_n factor (cf. [KR86, Ex. 6.9.16]). If $\mathcal{M}$ is of type II_1 then

$$\mathrm{tr}_N(\mathrm{F}_n\mathrm{E}\mathrm{F}_N) := \mathrm{tr}(\mathrm{F}_n\mathrm{E}\mathrm{F}_N)/\,\mathrm{tr}(\mathrm{F}_N) \quad \forall \mathrm{F}_n\mathrm{E}\mathrm{F}_N \in \mathrm{F}_N\mathcal{M}\mathrm{F}_N$$

is a normed trace on $\mathrm{F}_N\mathcal{M}\mathrm{F}_N$ since

$$\mathrm{tr}(\mathrm{F}_N) = \mathrm{tr}(\mathrm{I} - \sum_{k=1}^{N} \mathrm{E}_k) = 1 - \sum_{k=1}^{N} \mathrm{tr}(\mathrm{E}_k) \geq m_N.$$

If $\mathcal{M}$ is of type II_∞ then

$$\mathrm{tr}_N(\mathrm{F}_n\mathrm{E}\mathrm{F}_N) := \mathrm{tr}(\mathrm{F}_n\mathrm{E}\mathrm{F}_N) \quad \forall \mathrm{F}_n\mathrm{E}\mathrm{F}_N \in \mathrm{F}_N\mathcal{M}\mathrm{F}_N$$

is a trace on $\mathrm{F}_N\mathcal{M}\mathrm{F}_N$. The same argument as above now implies the existence of a projection $\mathrm{E}_N \in \mathrm{F}_N\mathcal{M}\mathrm{F}_N \subset \mathcal{M}$ such that $\mathrm{tr}_N(\mathrm{E}_N) = \mathrm{tr}(\mathrm{F}_N)^{-1}m_N \leq 1$, if $\mathcal{M}$ is finite, or $\mathrm{tr}_N(\mathrm{E}_N) = m_N$, if $\mathcal{M}$ is properly infinite. It follows that $\mathrm{tr}(\mathrm{E}_N) = m_N$ and $\mathrm{E}_N < \mathrm{F}_N \perp \mathrm{E}_k$ $(1 \leq k < N)$.

In the finite case the assertion is now proved by induction since

$$\mathrm{tr}(\sum_{k\in K} \mathrm{E}_k) = \sum_{k\in K} \mathrm{tr}(\mathrm{E}_k) = \sum_{k\in K} m_k = 1 = \mathrm{tr}(\mathrm{I})$$

and the trace is faithful.

Suppose now that $\mathcal{M}$ is of type II_∞ and that the family $(\mathrm{E}_k)_{k\in K}$ is constructed as above. If $\sum_{k\in K} \mathrm{E}_k = \mathrm{I}$ the proof is complete. Assume on the contrary that $\mathrm{F} := \sum_{k\in K} \mathrm{E}_k < \mathrm{I}$. Then $\mathrm{F} \sim \mathrm{I}$ since $\mathrm{tr}(\mathrm{F}) = \infty = \mathrm{tr}(\mathrm{I})$. There is hence a partial isometry $\mathrm{V} \in \mathcal{M}$ such that $\mathrm{V}\mathrm{V}^* = \mathrm{I}$ and $\mathrm{V}^*\mathrm{V} = \mathrm{F}$. Setting $\tilde{\mathrm{E}}_k := \mathrm{V}\mathrm{E}_k\mathrm{V}^*$, we get $\sum_k \tilde{\mathrm{E}}_k = \mathrm{I}$ and $\mathrm{tr}(\tilde{\mathrm{E}}_k) = \mathrm{tr}(\mathrm{V}\mathrm{E}_k\mathrm{V}^*) = m_k$. □

Theorem 4.3.12. *Let Δ be a positive invertible operator acting on a Hilbert space $\mathcal{H}$. Δ is a modular operator for a von Neumann algebra of type II_n, $n \in \{1, \infty\}$, with center (isomorphic to) $\mathcal{B}$ corresponding to a cyclic and separating vector which is diagonalizable with respect to the center if and only if Δ is an ∞-decomposable operator with respect to $\mathcal{B}$ with uniformly infinite central multiplicity which possesses multiplicative central spectrum of type II_n.*

Proof. If Δ is a modular operator then Δ is an ∞-decomposable operator with respect to $\mathcal{B}$ with uniformly infinite multiplicity which possesses multiplicative central spectrum of type I_n (see Remark 4.3.10.2).

We now prove the converse. Let $\mathcal{M}$ be an arbitrary type II_n von Neumann algebra acting on the Hilbert space $\mathcal{H}$ with center (isomorphic to) $\mathcal{B}$ and trace tr. The proof is divided into two steps. We first prove the factor case and then, using the direct integral decomposition, we generalize to the non-factor case.

1. Consider first the factor case, i.e. $\mathcal{B} = \mathcal{Z}(\mathcal{M}) = \mathbb{C}$ and $\mathcal{B}' = L(\mathcal{H})$. Proposition 4.3.11 implies that there exists a family of pairwise orthogonal projections $(\mathrm{E}_k)_{k\in K}$ such that $\mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k) = m_k$ and $\sum_{k\in K} \mathrm{E}_k = \mathrm{I}$. Setting

$$\mathrm{H}_0 := \sum_{k\in K} f_k \mathrm{E}_k$$

we get a positive invertible operator affiliated with $\mathcal{M}$ such that

$$\mathrm{tr}_{\mathcal{M}}(\mathrm{H}_0) = \sum_{k\in K} \mathrm{tr}_{\mathcal{M}}(f_k \mathrm{E}_k) = \sum_{k\in K} f_k m_k < \infty.$$

According to §3.2, if $\mathcal{M}$ is finite, or §3.3, if $\mathcal{M}$ is infinite, there is a cyclic and separating vector u_0 (the vector corresponding to $\mathrm{H}_0^{1/2}$ in the GNS-representation with respect to the trace) for $\mathcal{M}$ such that $\Delta_{u_0} = \mathrm{H}_0 \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0$ is the modular operator corresponding to u_0 with an appropriate modular conjugation J_0. The same calculations leading to (4.3.1) and Proposition 4.3.5 also yield $\Delta_{u_0} = \sum_{j\in J} g_j \hat{\mathrm{F}}_j$ where $\hat{\mathrm{F}}_j \in \mathcal{B}'$ and $\mathrm{tr}_{\mathcal{B}'}(\hat{\mathrm{F}}_j) = \infty$ $(j \in J)$. (2.3.2) now implies $\hat{\mathrm{F}}_j \sim \mathrm{F}_j$ in $\mathcal{B}' = L(\mathcal{H})$. The same argument as in the proof of Theorem 4.2.12 now proves the assertion in the factor case.

2. Suppose now that $\mathcal{M}$ is not a factor. We decompose $\mathcal{H}$ and $\mathcal{M}$ into direct integrals with respect to $\mathcal{B}$ as in the proof of Proposition 4.3.7. The central trace $\mathrm{tr}_{\mathcal{M}}$ can then be decomposed into traces tr_p on the factors $\mathcal{M}_p$, i.e.

$$\mathrm{tr} = \int_S \oplus\, \mathrm{tr}_p\, \mathrm{I}_p d\mu(p)$$

where I_p is the identity in $\mathcal{M}_p$. According to Proposition 4.3.11, we have families $(\mathrm{E}_p^k)_{k\in K}$ in $\mathcal{M}_p$ such that $\mathrm{tr}_p(\mathrm{E}_p^k) = m_k(p)$ for almost all $p \in S$. Setting $\mathrm{E}_k := \int_S \oplus \mathrm{E}_p^k d\mu(p)$, we get a family of projections in $\mathcal{M}$ such that $\mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k) = m_k$ for $k \in K$ and $\sum_{k\in K} \mathrm{E}_k = \mathrm{I}$. We define

$$\mathrm{H}_0 := \sum_{k\in K} f_k \mathrm{E}_k \eta \mathcal{M}$$

and proceed analogously to the first part since

$$\begin{aligned}
\mathrm{tr}(\mathrm{H}_0) &= \omega_\mu(\mathrm{tr}_{\mathcal{M}}(\mathrm{H}_0)) \\
&= \sum_{k\in K} \omega_\mu(f_k\, \mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k) \\
&= \sum_{k\in K} \omega_\mu(f_k m_k) < \infty.
\end{aligned}$$

□

The proof of Theorem 4.3.12 also implies the following

Corollary 4.3.13. *Let Δ be a modular operator for a type II_n algebra $\mathcal{M}$ corresponding to a cyclic and separating vector u_0. Suppose that $(m_k)_{k\in K}$ and $(f_k)_{k\in K}$ are two sequences fulfilling the prerequisites of Definition 4.3.9.2 with respect to Δ. Then there is a cyclic and separating vector u for $\mathcal{M}_0$ such that the corresponding modular operator is unitarily equivalent to Δ and $\Delta_u = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$. Moreover, $\mathrm{H} = \sum_k f_k \mathrm{E}_k$ is the decomposition of $\mathrm{H}\eta\mathcal{M}$ where $m_k = \mathrm{tr}_{\mathcal{M}}(\mathrm{E}_k)$.*

Remark 4.3.14. Note that it is not possible to distinguish between the different non-isomorphic algebras of type II with the help of the modular operators described in this section. In fact, assume that Δ is an operator which is ∞-decomposable with respect to an abelian algebra $\mathcal{A}$ with uniformly infinite central multiplicity and which possesses multiplicative spectrum of type II_1 (of type II_∞). Then there exist cyclic and separating vectors for *every* algebra of type II_1 (type II_∞) with center isomorphic to $\mathcal{A}$ such that the corresponding modular operator is unitarily equivalent to Δ (see the proof of Theorem 4.3.12).

We show in the next example that there exists a ∞-decomposable operator with uniformly infinite central multiplicity which possesses multiplicative central spectrum of type I_∞, II_1, and II_∞. It is thus a modular operator for a type I_∞, a type II_1, and a type II_∞ algebra.

Example 4.3.15. Let $0 < \lambda < 1$ be a positive number and $(\mathrm{F}_j)_{j\in\mathbb{Z}}$ be a sequence of pairwise orthogonal, infinite dimensional projections in an (infinite dimensional) Hilbert space $\mathcal{H}$ such that $\sum_j \mathrm{F}_j = \mathrm{I}$. The operator Δ defined by $\Delta := \sum_{j\in\mathbb{Z}} \lambda^j \mathrm{F}_j$ is then an ∞-decomposable operator with respect to $\mathbb{C}$ with uniformly infinite central multiplicity.

Let $m_k = 1$ and $f_k = \lambda^k(1-\lambda)$ for $k \in \mathbb{N}$. Then $\sum_{k\in\mathbb{N}} m_k = \infty$, $\sum_{k\in\mathbb{N}} m_k f_k = 1$, $\lambda^j = f_{k+j} f_k^{-1}$ for $k+j > 0$, and $\sum_{f_k f_l^{-1} = \lambda^j} m_k m_l = \infty$. These sequences demonstrate that Δ possesses multiplicative central spectrum of type I_∞. It is hence a modular operator for a type I_∞ factor according to Theorem 4.2.12.

The same sequences as in the preceding paragraph also demonstrate that Δ possesses multiplicative central spectrum of type II_∞ and, thus, it is a modular operator for a type II_∞ factor.

Setting $\hat{m}_k = (1-\lambda)\lambda^k$ and $\hat{f}_k = f_k = \lambda^k(1-\lambda)$ $(k \in \mathbb{N})$ we get $\sum_{k\in\mathbb{N}} \hat{m}_k = 1$ and $\sum_{k\in\mathbb{N}} \hat{m}_k \hat{f}_k = (1-\lambda)^2/(1-\lambda^2) < \infty$. This implies that Δ also possesses multiplicative central spectrum of type II_1. Hence, it is a modular operator for a type II_1 factor.

4.3.2 Remarks on the General Case

Whereas the results for the type I case and for the diagonalizable case of type II algebras are quite satisfactory it is unclear if and how these results can be extended to general cyclic and separating vector for type II algebras. In this subsection we present some examples and minor results which illustrate the problems occurring in the general case.

We showed in the type I case that every modular operator corresponding to a cyclic and separating vector is a decomposable operator with respect to an abelian algebra. In the type II case this is not longer true (Example 4.3.3). Moreover, a modular operator which corresponds to a diagonalizable vector can also be the modular operator corresponding to a vector which is not diagonalizable with respect to the center, as the following example will show.

Example 4.3.16. Let $\mathcal{R} = \mathcal{R}(\mathcal{A}, \alpha)$ be the type II_1 factor defined in Example 2.2.6 (cf. also Example 4.3.3) and $f := \exp \in \mathcal{A}$. Then $\mathrm{tr}(\Phi(f)) = \int f d\lambda < \infty$ and $\Phi(f)$ is not diagonalizable with respect to the center (see Example 4.3.3). If J is the modular conjugation defined by $\mathrm{J}\mathrm{A}u_{\mathrm{tr}} = \mathrm{A}^* u_{\mathrm{tr}}$ with $u_{\mathrm{tr}} = [\delta_{p,0} 1] \in \mathcal{H} \otimes L_2(G)$ then

$$\Delta = \Phi(f)\mathrm{J}\Phi(f)\mathrm{J} = [\delta_{p,q} f \mathrm{U}(p) f^{-1} \mathrm{U}(p)^*]_{p,q} =: [\delta_{p,q} g_p]_{p,q}$$

is the modular operator corresponding to $u_0 := \Phi(f)^{1/2} u_{\mathrm{tr}}$. Since $\mathrm{ad}\,\mathrm{U}(p) = \alpha_p$ are the translations by p in $\mathcal{A}$ the functions g_p are

$$g_p(x) = \begin{cases} \exp(p) & \text{if } 0 \le x - p < 1 \\ \exp(p-1) & \text{if } -1 < x - p < 0 \end{cases}.$$

Hence, the eigenvalues of Δ are $\{e^r | r \in \mathbb{Q}, -1 < r < 1\}$, and the corresponding spectral projections are $\mathrm{F}_r = [\delta_{p,q} \delta_{pr} \chi_r]$, if $r \ge 0$, or $\mathrm{F}_r = [\delta_{p,q} \delta_{p(r+1)} \chi_r]$, if $r < 0$, where χ_r is the characteristic function of the set $\{0 \le x < 1 | 0 \le x - r < 1\}$. This implies that Δ is a ∞-decomposable operator with respect to $\mathbb{C}$ with uniformly infinite central multiplicity. Set $f_r := e^r$ for $r \in \mathbb{Q}$ with $0 \le r < 1$ and let $(m_r)_{r \in \mathbb{Q}}$ be a sequence of positive numbers such that $\sum_r m_r = 1$. Using the sequences (f_r) and (m_r) we deduce that Δ possesses also multiplicative central spectrum of type II_1. It is hence the modular operator corresponding to a cyclic and separating vector which is diagonalizable with respect to the center.

The last example shows that it is by no means trivial to conclude from the spectral structure of the modular operator to the spectral structure of the operator which generates the modular operator. The only positive results of this type are the following statements which we treat only in the factor case.

Lemma 4.3.17. *Let Δ be the modular operator for a type II_n factor corresponding to a cyclic and separating vector, and let F_j $(j \in J)$ be the eigenprojections of Δ.*

(i) If $\sum_{j \in J} \mathrm{F}_j \ne \mathrm{I}$ then the generating operator (the operator corresponding to the cyclic and separating vector) is not diagonalizable.

(ii) If the spectrum of Δ is a countable set then the generating operator (the operator corresponding to the cyclic and separating vector) is diagonalizable.

Proof. Let $u_0 \in \mathcal{H}$ be the cyclic and separating vector of the modular operator Δ and H_0 be the corresponding positive operator affiliated with $\mathcal{M}$. We denote the spectral measure of H_0 by E.

(i) If u_0 was diagonalizable then $\sum_{j \in J} \mathrm{F}_j = \mathrm{I}$ according to (4.3.1), which contradicts the assumption.

(ii) If the spectrum of Δ is a countable set then the spectrum of H_0 is also a countable set (see Corollary 4.1.3). Let $(f_k)_{k\in K}$ be the spectral points of H_0 and $\mathrm{E}_k = \mathrm{E}(\{f_k\}) \in \mathcal{M}$ the corresponding spectral projections $(k \in K)$. If $\sum_{k\in K} \mathrm{E}_k \neq \mathrm{I}$ then there would be a Borel set $M \subset \mathbb{R} \setminus \{f_k | k \in K\}$ such that $\mathrm{E}(M) \neq 0$. But then $M \cap \sigma(\mathrm{H}_0) \neq \emptyset$, which contradicts the fact that $\sigma(\mathrm{H}_0) = \{f_k | k \in K\}$. Hence, $\mathrm{H}_0 = \sum_{k\in K} f_k \mathrm{E}_k$ with $\sum_k \mathrm{E}_k = \mathrm{I}$. □

To formulate the next result we must first introduce some notions. A *maximal abelian subalgebra* $\mathcal{A}$ in a von Neumann algebra $\mathcal{M}$ is an abelian algebra maximal in the set of abelian subalgebras of $\mathcal{M}$. A maximal abelian subalgebra $\mathcal{A}$ is called a *Cartan subalgebra* in $\mathcal{M}$ if the *normalizer* $\mathcal{N}_{\mathcal{M}}(\mathcal{A}) := \{\mathrm{U} \in \mathcal{U}(\mathcal{M}) | \operatorname{ad}\mathrm{U}(\mathcal{A}) = \mathcal{A}\}$ of $\mathcal{A}$ in $\mathcal{M}$ generates $\mathcal{M}$, i.e. if $(\mathcal{N}_{\mathcal{M}}(\mathcal{A}))'' = \mathcal{M}$.

Let now $\mathcal{M}$ be a hyperfinite algebra, and let $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$ be two Cartan subalgebras of $\mathcal{M}$. Connes, Feldman, and Weiss showed in [CFW81] that $\mathcal{A}$ and $\mathcal{B}$ are conjugated by an automorphism of $\mathcal{M}$, i.e. there is an automorphism $\beta \in \operatorname{aut}(\mathcal{M})$ such that $\mathcal{A} = \beta(\mathcal{B})$ (see also [Pop85] for the type II_1 factor case). Since the factor $\mathcal{R} = \mathcal{R}(\mathcal{A}, \alpha)$ constructed in Example 2.2.6 is the (unique) hyperfinite factor of type II_1 (see e.g. [KR86, 12.4.22]) and $\mathcal{A} \subset \mathcal{R}$ is a Cartan subalgebra of $\mathcal{R}$ ($\mathcal{R} = (\mathcal{A} \cup \{\mathrm{U}(p)\})''$ where $\{\mathrm{U}(p)\} \subset \mathcal{N}_{\mathcal{M}}(\mathcal{A})$) we can generalize Example 4.3.16 to

Lemma 4.3.18. *Let u_0 be a cyclic and separating vector for the hyperfinite type II_1 factor $\mathcal{M}$ and let $\mathrm{H}_0 \eta \mathcal{M}$ be the positive invertible operator which generates the modular operator corresponding to u_0. Suppose that H_0 is affiliated with a Cartan subalgebra $\mathcal{A}$ of $\mathcal{M}$.*

Assume that $\mathrm{H}_0 = \mathrm{M}_f \eta \mathcal{A}$ for a positive measurable function f. Then the modular operator corresponding to u_0 is

$$\Delta = [\delta_{p,q} f \mathrm{U}(p) f^{-1} \mathrm{U}(p)^*]_{p,q} =: [\delta_{p,q} g_p]_{p,q} \tag{4.3.3}$$

where $\mathrm{U}(p) \in \mathcal{N}_{\mathcal{M}}(\mathcal{A})$ $(p \in \mathbb{Q})$ such that $\mathcal{M} = (\mathcal{A} \cup \{\mathrm{U}(p)\})''$.

Proof. Since the hyperfinite factor of type II_1 is unique (up to isomorphisms) and the Cartan subalgebras of hyperfinite algebras are unique (up to conjugacy) we can assume without loss of generality that we are in the situation of Example 4.3.16 where $f \eta \mathcal{A}$ is arbitrary with $\int f d\mu < \infty$. Hence the assertion follows with the same calculations as in Example 4.3.16. □

The spectrum of the modular operator Δ described in Lemma 4.3.18 is

$$\sigma(\Delta) = \overline{\bigcup_{p\in\mathbb{Q}} \sigma(g_p)} = \overline{\bigcup_{p\in\mathbb{Q}} \sigma(f f^{-1}(\cdot - p))}. \tag{4.3.4}$$

Conversely, every positive operator Δ affiliated with $\oplus_p \mathcal{A}$ for a diffuse abelian algebra $\mathcal{A}$ such that Δ has the form (4.3.3) for an $f \eta \mathcal{A}$ with $\int f d\mu < \infty$ is a modular operator for the hyperfinite type II_1 factor.

Remark 4.3.19. 1. Lemma 4.3.18 and (4.3.4) are really generalizations of the results of §4.3.1 for the hyperfinite type II_1 factor. In fact, every operator affiliated with the hyperfinite type II_1 factor which has pure point spectrum is affiliated with a Cartan subalgebra, as the following considerations show:

Let $\mathrm{H}_0 = \sum_k f_{k\in K}\mathrm{E}_k \eta \mathcal{R} = \mathcal{R}(\mathcal{A}, \alpha)$ be an operator with pure point spectrum, and let $m_k := \mathrm{tr}(\mathrm{E}_k)$ $(k \in K)$ be the multiplicities of the eigenvalues of H_0. Since $\mathcal{A} = L_\infty([0,1)\,, \lambda)$ and $\sum_k m_k = 1$ there is a partition $(M_k)_{k\in K}$ of the unit interval into disjoint intervals M_k such that $\lambda(M_k) = m_k$. Let now $\mathrm{M}_k \in \mathcal{A}$ be the characteristic functions of the intervals M_k. Then $\mathrm{tr}(\mathrm{M}_k) = m_k = \mathrm{tr}(\mathrm{E}_k)$ and, accordingly, there is a unitary $\mathrm{U} \in \mathcal{U}(\mathcal{M})$ such that $\mathrm{M}_k = (\mathrm{ad}\,\mathrm{U})(\mathrm{E}_k)$ for all k. This implies that $\mathrm{H}_0 = \sum_k f_k \mathrm{E}_k = \sum_k f_k \mathrm{U}^* \mathrm{M}_k \mathrm{U}$ is affiliated with the Cartan subalgebra $\mathrm{U}^*\mathcal{A}\mathrm{U}$ of $\mathcal{R}$.

2. Unfortunately, not every positive operator is affiliated with a Cartan subalgebra also in the hyperfinite case. This follows from the fact that the hyperfinite II_1 factor contains so-called singular maximal abelian subalgebras (see e.g. [Dix54, Puk56, Tau65, Pop83]) (*singular maximal abelian subalgebras* are maximal abelian subalgebras $\mathcal{A}$ of a von Neumann algebra $\mathcal{M}$ whose normalizer in $\mathcal{M}$ is a subset of $\mathcal{A}$) . Since a singular maximal abelian subalgebra $\mathcal{A}$ of the hyperfinite II_1 factor is diffuse it is isomorphic to $L_\infty([0,1)\,, \lambda)$. Hence, there is a L_1 function in $\mathcal{A}$ generating $\mathcal{A}$ as a von Neumann algebra (e.g. the exponential function). This function can not be affiliated with a Cartan subalgebra since otherwise there would be an abelian algebra $\mathcal{B} \supsetneqq \mathcal{A} = \{f\}''$ which contradicts the maximality of $\mathcal{A}$.

3. For non-hyperfinite factors Lemma 4.3.18 is in general not true since there is an example of a II_1 factor with two non-conjugate Cartan subalgebras ([CJ82]).

4.4 Type III Factors

Since we showed in §3.4.1 that the type III_λ case $(0 \le \lambda < 1)$ can essentially be reduced to the type II_∞ case this section uses many results of §4.3. In this section $\mathcal{M}$ is always a type III_λ $(0 \le \lambda < 1)$ factor acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector $u_0 \in \mathcal{H}$.

Let $(\mathcal{N}, \theta, \mathrm{tr})$ be the discrete decomposition corresponding to the cyclic and separating vector u_0 where $\mathcal{N}$ is a type II_∞ von Neumann algebra. The weight $\tau := \hat{\mathrm{tr}}$ on $\mathcal{M}$ is the dual of the trace tr on $\mathcal{N}$ (see Proposition 3.4.1). Let further $\mathrm{T}_0 \in \mathcal{N}$ be the invertible operator corresponding to u_0 with polar decomposition $\mathrm{T}_0 = \mathrm{H}_0^{1/2}\mathrm{V}$ such that $\Delta_0 = \mathrm{H}_0 \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0 \Delta_{\tilde{\mu}_0}$ and $\mathrm{J}_0 = \mathrm{V}\Delta_{\tilde{\mu}_0}\mathrm{V}^*$ are the modular objects of u_0. We call u_0 *diagonalizable with respect to the center* if $\mathrm{H}_0 \in \mathcal{N}$ is diagonalizable with respect to the center $\mathcal{Z}(\mathcal{N})$ of $\mathcal{N}$ (cf. Definition 4.3.1).

From now on we make the assumption that u_0 is diagonalizable with respect to the center, i.e. there are a family $(f_k)_{k\in\mathbb{N}}$ of positive functions $f_k \eta \mathcal{Z}(\mathcal{N})$ and a family $(\mathrm{E}_k)_{k\in\mathbb{N}}$ of projections $\mathrm{E}_k \in \mathcal{N}$ such that $\mathrm{H}_0 = \sum_{k\in\mathbb{N}} f_k \mathrm{E}_k$ and

$$m_k := \mathrm{tr}_{\mathcal{N}}(\mathrm{E}_k)\eta \mathcal{Z}(\mathcal{M}). \tag{4.4.1}$$

Furthermore, we assume without loss of generality that $\mathrm{V} = \mathrm{I}$ (choose $\tilde{\mu}_0(\cdot\mathrm{V})$ instead of $\tilde{\mu}_0$ for the cyclic and separating generalized vector generating τ, see §3.4.1). With the notation introduced in §3.4 we have $\mathrm{E}_k = [\delta_{n,m}\mathrm{E}_k]_{n,m}$ and $\mathrm{J}_0\mathrm{E}_k\mathrm{J}_0 = [\delta_{n,m}\mathrm{U}(n)\mathrm{J}\mathrm{E}_k\mathrm{J}\mathrm{U}(n)^*]$ (cf. Corollary 3.4.5). Here we use slightly inconsistently the same symbol E_k for projections in $\mathcal{N}$ as well as for the image of E_k under the isomorphism from $\mathcal{R}(\mathcal{N},\theta)$ into $\mathcal{M}$.

Moreover, $\Delta_{\tilde{\mu}_0} = [\delta_{n,m}\mathrm{U}(n)\mathrm{A}_n\mathrm{U}(n)^*]_{n,m}$ (cf. Proposition 3.4.3) where A_n is the unique positive invertible operator affiliated with $\mathcal{Z}(\mathcal{N})$ such that

$$\mathrm{tr}_{\mathcal{N}} \circ \theta^n = (\mathrm{tr}_{\mathcal{N}})_{\mathrm{A}_n} \quad (n \in \mathbb{Z}) \tag{4.4.2}$$

for $\theta^n = \mathrm{ad}\,\mathrm{U}(n)$.

Similarly to the preceding sections the following results are more obvious if we consider the type III_λ case $(0 < \lambda < 1)$ because $\mathcal{N}$ is then a type II_∞ factor and the operators A_n are merely numbers λ^n such that many problems concerned with domains of definition vanish.

The modular operator Δ_0 corresponding to u_0 now has the following decomposition (cf. Theorem 3.4.6):

$$\begin{aligned}
\Delta_0 &= \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\Delta_{\tilde{\mu}_0} \\
&= \sum_{k,l\in\mathbb{N}} (f_k\mathrm{E}_k)\mathrm{J}_0(f_l^{-1}\mathrm{E}_l)\mathrm{J}_0\Delta_{\tilde{\mu}_0} \\
&= \sum_{k,l\in\mathbb{N}} [\delta_{n,m}f_k\mathrm{E}_k\mathrm{U}(n)\mathrm{J}f_l^{-1}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^*\mathrm{U}(n)\mathrm{A}_n\mathrm{U}(n)^*]_{n,m} \\
&= \sum_{k,l\in\mathbb{N}} [\delta_{n,m}f_k\mathrm{E}_k\mathrm{U}(n)f_l^{-1}\mathrm{A}_n\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^*]_{n,m} \\
&= \sum_{k,l\in\mathbb{N}} [\delta_{n,m}f_k\mathrm{U}(n)f_l^{-1}\mathrm{A}_n\mathrm{U}(n)^*\mathrm{E}_k\mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^*]_{n,m} \\
&=: \sum_{j\in\mathbb{N}}[\delta_{n,m}g_j^n\mathrm{F}_j^n]_{n,m} \\
&=: \sum_{j\in\mathbb{N}} g_j\mathrm{F}_j
\end{aligned} \tag{4.4.3}$$

where every function g_j^n equals one of the products $f_k\mathrm{U}(n)f_l^{-1}\mathrm{A}_n\mathrm{U}(n)^*$, $g_j^n \neq g_i^n$ for $i \neq j$ and at least one $n \in \mathbb{Z}$, and the F_j^n are pairwise orthogonal $(n \in \mathbb{Z})$. In addition, $g_j = [\delta_{n,m}g_j^n]$ and $\mathrm{F}_j = [\delta_{n,m}\mathrm{F}_j^n]$.

The next lemma is the analogue of Lemma 4.2.6 and Lemma 4.3.4:

Lemma 4.4.1. *With the notations introduced above, Δ_0 has the spectrum*

$$\sigma(\Delta_0) = \overline{\bigcup_{j\in\mathbb{N},n\in\mathbb{Z}} \sigma(g_j^n)} = \overline{\bigcup_{k,l\in\mathbb{N},n\in\mathbb{Z}} \sigma(f_k\mathrm{U}(n)f_l^{-1}\mathrm{A}_n\mathrm{U}(n)^*)}. \tag{4.4.4}$$

Proof. Note first that $\mathrm{C}_{\mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^*} = \mathrm{U}(n)\mathrm{C}_{\mathrm{E}_l}\mathrm{U}(n)^*$ and $\mathrm{supp}(f_l) = \mathrm{C}_{\mathrm{E}_l}$ for $l \in \mathbb{N}$. Hence, $\mathrm{E}_k\mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^* \neq 0$ if and only if the intersection of the supports of f_k and $\mathrm{U}(n)f_l\mathrm{U}(n)^*$ is a non-null set (cf. Lemma 4.2.6). The assertion now follows directly from (4.4.3) and

$$\mathrm{F}_j^n = \sum_{f_k \mathrm{U}(n) f_l^{-1} \mathrm{A}_n \mathrm{U}(n)^* = g_j^n} (\mathrm{E}_k \mathrm{U}(n) \mathrm{J} \mathrm{E}_l \mathrm{J} \mathrm{U}(n)^*) \neq 0.$$

□

In the next proposition we collect some further properties of the projections F_j appearing in (4.4.3).

Proposition 4.4.2. *Let $\mathcal{N}$, g_j, and F_j be as above, and set*

$$\mathcal{B} := \bigoplus_{n \in \mathbb{Z}} \mathcal{Z}(\mathcal{N}) = [\delta_{m,n} \mathcal{Z}(\mathcal{N})].$$

Each projection F_j commutes with $\mathcal{B}$ and has central carrier in $\mathcal{B}$ equal to the support of the corresponding g_j.

Proof. Note first that $\mathrm{E}_k \in \mathcal{N}$ $(k \in \mathbb{N})$ obviously commutes with $\mathcal{Z}(\mathcal{N})$. If $\mathrm{B} \in \mathcal{Z}(\mathcal{N})$ then

$$\begin{aligned} \mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^*\mathrm{B} &= \mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\underbrace{\mathrm{U}(n)^*\mathrm{B}\mathrm{U}(n)}_{\in \mathcal{Z}(\mathcal{N})}\mathrm{U}(n)^* \\ &= \mathrm{U}(n)\mathrm{U}(n)^*\mathrm{B}\mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^* \\ &= \mathrm{B}\mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^*. \end{aligned}$$

Accordingly, $\mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^*$ commutes with $\mathcal{Z}(\mathcal{N})$ for all $n \in \mathbb{N}$, $l \in \mathbb{N}$. Being a sum of projections $\mathrm{E}_k\mathrm{U}(n)\mathrm{J}\mathrm{E}_l\mathrm{J}\mathrm{U}(n)^*$, the projection F_j^n commutes with $\mathcal{Z}(\mathcal{N})$ and $\mathrm{F}_j = [\delta_{n,m}\mathrm{F}_j^n]$ commutes with $\mathcal{B} = [\delta_{m,n}\mathcal{Z}(\mathcal{N})]$.

The assertion regarding the central carriers follows from the corresponding assertion in Proposition 4.3.5 since every F_j is the direct sum of projections in $\mathcal{Z}(\mathcal{N})'$. □

Let now $\mathrm{tr}_{\mathcal{B}'}$ be the central trace on the commutant $\mathcal{B}'$ of $\mathcal{B}$ uniquely determined by the condition $\mathrm{tr}_{\mathcal{B}'}(\mathrm{E}) = \mathrm{I}$ for all abelian projection $\mathrm{E} \in \mathcal{B}'$ with central carrier I. As in §4.2 and §4.3 we set $n_j := \mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j)$ $(j \in \mathbb{N})$ and call it the *central multiplicity of* F_j *in* $\mathcal{B}'$. The functions m_k defined in (4.4.1) are again the *central multiplicities of* E_k *in* $\mathcal{M}$.

Proposition 4.4.3. *With the notations introduced above, we have*

$$n_j = \mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j) = \infty \cdot \mathrm{I}$$

for all $j \in \mathbb{N}$.

Proof. The assertion follows from the corresponding assertion in Proposition 4.4.3 since every projection F_j is the direct sum of projections in $\mathcal{Z}(\mathcal{N})'$. □

Remark 4.4.4. Lemma 4.4.1 and Proposition 4.4.2 imply that a modular operator for a type III_λ factor $(0 \leq \lambda < 1)$ corresponding to a cyclic and separating vectors which is diagonalizable with respect to the center is a ∞-decomposable operator with respect to $\mathcal{B}$ with uniformly infinite central multiplicity in the sense of Definition 4.3.9.

The analogue of Definition 4.2.10 and Definition 4.3.9 is the following

Definition 4.4.5. Let Δ be a positive invertible operator acting on a Hilbert space $\mathcal{H}$, and let $\mathcal{B}$ be one of the following algebras: $\mathcal{A}_\infty = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}$ or $\bigoplus_{n\in\mathbb{Z}} \mathcal{A}_c$ (see Theorem 4.2.2 for notations). Suppose that $\Delta = \sum_{j\in\mathbb{N}} g_j \mathrm{F}_j$ is ∞-decomposable with respect to $\mathcal{B}$ with uniformly infinite central multiplicity.

Δ possesses *multiplicative central spectrum of type* III_λ *for* $0 < \lambda < 1$ *(of type* III_0*)* if there are sequences $(f_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ of positive reals f_k, m_k (positive elements f_k, m_k affiliated with $\mathcal{A}_c$) fulfilling the following properties:

- $\sum_{k\in\mathbb{N}} m_k = \infty$,
- $\sum_{k\in\mathbb{N}} m_k f_k < \infty$ $(\sum_{k\in\mathbb{N}} \omega_\mu(m_k f_k) < \infty)$,
- for every $j \in \mathbb{N}$ there exists at least one pair $(k,l) \in \mathbb{N}^2$ with $g_j^n = f_k f_l^{-1} \lambda^n$ $(g_j^n = f_k \mathrm{U}(n) f_l^{-1} \mathrm{A}_n \mathrm{U}(n)^*)$ where $g_j = [g_j^n]$ and the operators A_n and $\mathrm{U}(n)$ are defined in (4.4.2)).

Remark 4.4.6.
1. Note that the definition depends on the factor $\mathcal{M}$ used to define the operators A_n and $\mathrm{U}(n)$ in the type III_0 case whereas in the type III_λ case $(0 < \lambda < 1)$ the relation $\mathrm{A}_n = \lambda^n$ $(n \in \mathbb{Z})$ holds for all factors and the unitaries $\mathrm{U}(n)$ disappear.
2. The preceding considerations show that the modular operators for type III_λ (type III_0) factors corresponding to cyclic and separating vectors which are diagonalizable with respect to the center are operators with multiplicative central spectrum of type III_λ (type III_0).

Theorem 4.4.7. *Let Δ be a positive invertible operator acting on a Hilbert space $\mathcal{H}$. Δ is a modular operator for a von Neumann factor of type III_λ, $0 < \lambda < 1$, (of type III_0) corresponding to a cyclic and separating vector which is diagonalizable with respect to the center if and only if Δ is an ∞-decomposable operator with respect to $\mathcal{A}_\infty$ $(\bigoplus_{n\in\mathbb{Z}} \mathcal{A}_c)$ with uniformly infinite central multiplicity which possesses multiplicative central spectrum of type III_λ (of type III_0).*

Proof. If Δ is a modular operator then Δ is an ∞-decomposable operator with respect to $\mathcal{B}$ with uniformly infinite multiplicity which possesses multiplicative central spectrum of type I_n (see Remark 4.3.10.2).

We now prove the converse. Let $\mathcal{M}$ be a type III_λ (type III_0) von Neumann factor acting on the Hilbert space $\mathcal{H}$, and let $(\mathcal{N}, \theta, \mathrm{tr})$ be a discrete decomposition of $\mathcal{M}$ such that $\mathcal{N}$ is a type II_∞ von Neumann algebra with center $\mathbb{C}$ $(\mathcal{A}_c)$ and $\mathrm{tr} \circ \theta^n = \lambda^n \,\mathrm{tr}$ $(\mathrm{tr} \circ \theta^n = \mathrm{tr}_{\mathrm{A}_n})$ $(n \in \mathbb{Z})$.

As in the proof of Theorem 4.3.12, there exists a family of pairwise orthogonal projections $(\mathrm{E}_k)_{k\in\mathbb{N}}$ in $\mathcal{N}$ such that $\mathrm{tr}_{\mathcal{N}}(\mathrm{E}_k) = m_k$ and $\sum_{k\in\mathbb{N}} \mathrm{E}_k = \mathrm{I}$. Setting

$$\mathrm{H}_0 := \sum_{k\in\mathbb{N}} f_k \mathrm{E}_k,$$

we get a positive invertible operator affiliated with $\mathcal{N}$ such that

$$\mathrm{tr}(\mathrm{H}_0) = \sum_{k\in\mathbb{N}} \mathrm{tr}(f_k\mathrm{E}_k) = \sum_{k\in\mathbb{N}} \omega_\mu(f_k) m_k < \infty$$

(cf. again the proof of Theorem 4.3.12). According to §3.3, there is a cyclic and separating vector v_0 for $\mathcal{N}$ and, according to §3.4, a cyclic and separating vector $u_0 = v_0 \otimes x_0 \in \mathcal{H}$ for $\mathcal{M}$ such that $\Delta_{u_0} = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\Delta_{\tilde{\mu}_0}$ is the modular operator corresponding to u_0 with an appropriate modular conjugation J_0. The same calculations as in (4.4.3) and in the proof of Proposition 4.4.2 yield $\Delta_{u_0} = \sum_{j\in\mathbb{N}} g_j\hat{\mathrm{F}}_j$ where $\hat{\mathrm{F}}_j \in \mathcal{B}'$ and $\mathrm{tr}_{\mathcal{B}'}(\hat{\mathrm{F}}_j) = \infty\cdot\mathrm{I}$ $(j \in \mathbb{N})$. The same argument as in the proof of Theorem 4.3.12 now proves the assertion. □

The proof of Theorem 4.4.7 also implies the following

Corollary 4.4.8. *Let Δ be a modular operator for a type III_λ factor $\mathcal{M}$ corresponding to a cyclic and separating vector u_0. Suppose that $(m_k)_{k\in K}$ and $(f_k)_{k\in K}$ are two sequences fulfilling the prerequisites of Definition 4.4.5 with respect to Δ. Then there is a cyclic and separating vector u for $\mathcal{M}_0$ such that the corresponding modular operator is unitarily equivalent to Δ and $\Delta_u = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0\Delta_{\tilde{\mu}_0}$. Moreover, $\mathrm{H} = \sum_k f_k\mathrm{E}_k$ is the decomposition of $\mathrm{H} \in \mathcal{N}$ where $m_k = \mathrm{tr}_{\mathcal{N}}(\mathrm{E}_k)$.*

Remark 4.4.9. A result analogous to that presented here also holds in the circumstances described in §3.4.2 for the type III_1 case. The only changes to make are the substitution of $\mathbb{Z}$ by $\mathbb{Z}^2$ and of A_n $(n \in \mathbb{Z})$ by $\lambda_1^{z_1}\lambda_2^{z_2}$ $(z_1, z_2 \in \mathbb{Z})$.

Chapter 5

Inverse Problems in Modular Theory

The investigation of inverse problems in modular theory is motivated by physical applications of modular theory (cf. §2.4). This motivation is studied in the first section in greater detail and the formulations of the resulting mathematical problems are given. The second section is devoted to general properties of the inverse problems and a first simple class of solutions. In the last section we introduce an equivalence relation on the set of solutions of one (exemplary) inverse problem and consider a second simple class of solutions.

The idea of considering inverse problems in modular theory is due to Wollenberg (see [Wol92] and [Wol97], cf. also [BW01]). In the factor case most of the results of §5.2 were proved in [Wol97] and [BW01] whereas the general results are new. The equivalence relation was introduced by the author (cf. [Bol00a] and [Bol00b]) whereas the second simple class was introduced by Wollenberg for finite type I factors (see [Wol97] and [BW01]).

5.1 Motivation and Formulation

To motivate the investigation of inverse problems in modular theory we first recall the physical application presented in §2.4. The net $(\mathcal{M}(\mathcal{O}))_{\mathcal{O}}$ is often a so-called *simple causal net* (see e. g. [BW92, 7.3.6] or [Wol92] for the definition). In particular, the whole net of algebras is determined by one of the local algebras and the representation of the symmetry group. If we assume duality in the example of §2.4, the local algebra is the algebra of the wedge $\mathcal{W}$ and the symmetry group is the Poincaré group. Under suitable assumptions the wedge algebra $\mathcal{M}(\mathcal{W})$ is isomorphic to the hyperfinite III_1 factor (see e. g. [BW92] and §2.4). Furthermore, the Bisognano-Wichmann-Theorem (Theorem 2.4.2) states that the modular operator Δ for the wedge algebra corresponding to the (cyclic and separating) vacuum is the generator of the Lorentz boosts in the wedge and the modular conjugation is the product of the PCT-operator with a rotation.

Moreover, one can construct a free (linear) quantum field theory $(\mathcal{M}_0(\mathcal{O}))_{\mathcal{O}}$ for the same representation of the Poincaré group and the same vacuum which also satisfies the mentioned assumptions. The corresponding wedge algebra $\mathcal{M}_0(\mathcal{W})$ is hence the hyperfinite III_1 factor (it is therefore unitarily equivalent to $\mathcal{M}(\mathcal{W})$), the whole net is determined by the wedge algebra and the representation of the Poincaré group, and the modular operator for $\mathcal{M}_0(\mathcal{W})$ corresponding to the vacuum is the same as that for $\mathcal{M}(\mathcal{W})$ (for details we refer e. g. to [BW92]). In addition, the modular conjugations of the two wedge algebras differ only by a unitary commuting with the representation of the Poincaré group (see [Wol92]).

This observation essentially reduces the study of the possible quantum field theories for a fixed vacuum and a fixed representation of the Poincaré group to the study of the following problem (see [Wol92]):

Problem 5.1.1. Find all unitaries $\mathrm{V} \in \mathcal{U}(\mathcal{H})$ such that

1. Ω is cyclic and separating for $\mathrm{V}\mathcal{M}_0\mathrm{V}^*$ ($\mathcal{M}_0 := \mathcal{M}_0(\mathcal{W})$).
2. $(\mathrm{V}\mathcal{M}_0\mathrm{V}^*, \Omega)$ has the same modular objects as $(\mathcal{M}_0, \Omega)$.
3. $\mathrm{ad}\,\mathrm{V}^*\mathrm{U}(\Lambda, a)\mathrm{V}$ is an endomorphism of $\mathcal{M}_0$ for all elements (Λ, a) of the Poincaré group mapping the wedge into itself.

Properties 1 and 2 motivated Wollenberg in [Wol92, Wol97, Wol98] to pose and investigate the following inverse problems:

Problem 5.1.2 (Inverse problem for the modular objects of $(\mathcal{M}_0, u_0)$). Let $\mathcal{M}_0$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $u_0 \in \mathcal{H}$ be a cyclic and separating unit vector for $\mathcal{M}_0$. Suppose (Δ_0, J_0) are the modular objects of $(\mathcal{M}_0, u_0)$. Characterize all von Neumann algebras $\mathcal{M}$ acting on the same Hilbert space $\mathcal{H}$ which satisfy the following properties:

1. $\mathcal{M}$ is isomorphic to $\mathcal{M}_0$,
2. u_0 is also a cyclic and separating vector for $\mathcal{M}$,
3. (Δ_0, J_0) are the modular objects of $(\mathcal{M}, u_0)$.

The set of all von **N**eumann **A**lgebras satisfying 1-3 is denoted by

$$NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0).$$

Remark 5.1.3. Note that we consider the von Neumann algebra $\mathcal{M}_0$ as a concrete algebra of operators acting on a concrete Hilbert space and not as an isomorphism class of algebras. Thus, we distinguish the representing algebras of the isomorphism class of $\mathcal{M}_0$ such that it makes sense to speak of the modular objects of $\mathcal{M}_0$ and to look only for von Neumann algebras $\mathcal{M}$ isomorphic to $\mathcal{M}_0$. The same remark also applies to the following inverse problems. At the end of §5.2 we will show that more general inverse problems can sometimes be reduced to Problem 5.1.2.

Problem 5.1.4 (Inverse problem for the modular operator of $(\mathcal{M}_0, u_0)$). Let $\mathcal{M}_0$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $u_0 \in \mathcal{H}$ be a cyclic and separating unit vector for $\mathcal{M}_0$. Suppose Δ_0 is the modular operator of $(\mathcal{M}_0, u_0)$. Characterize all von Neumann algebras $\mathcal{M}$ acting on the same Hilbert space $\mathcal{H}$ which satisfy the following properties:

1. $\mathcal{M}$ is isomorphic to $\mathcal{M}_0$,
2. u_0 is a cyclic and separating vector for $\mathcal{M}$,
3. Δ_0 is the modular operator of $(\mathcal{M}, u_0)$.

The set of all von **N**eumann **A**lgebras satisfying 1-3 is denoted by

$$NA(\Delta_0, u_0; \mathcal{M}_0).$$

The following problems are closely related to Problem 5.1.2 and Problem 5.1.4.

Problem 5.1.5 (Inverse problem for the modular objects of $\mathcal{M}_0$). Let $\mathcal{M}_0$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $u_0 \in \mathcal{H}$ be a cyclic and separating unit vector for $\mathcal{M}_0$. Suppose (Δ_0, J_0) are the modular objects of $(\mathcal{M}_0, u_0)$. Characterize all von Neumann algebras $\mathcal{M}$ acting on the same Hilbert space $\mathcal{H}$ which satisfy the following properties:

1. $\mathcal{M}$ is isomorphic to $\mathcal{M}_0$,
2. there exists a cyclic and separating unit vector u for $\mathcal{M}$,
3. (Δ_0, J_0) are the modular objects of $(\mathcal{M}, u)$.

The set of all von **N**eumann **A**lgebras satisfying 1-3 is denoted by

$$NA(\Delta_0, \mathrm{J}_0; \mathcal{M}_0).$$

Problem 5.1.6 (Inverse problem for the modular operator of $\mathcal{M}_0$). Let $\mathcal{M}_0$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $u_0 \in \mathcal{H}$ be a cyclic and separating unit vector for $\mathcal{M}_0$. Suppose Δ_0 is the modular operator of $(\mathcal{M}_0, u_0)$. Characterize all von Neumann algebras $\mathcal{M}$ acting on the same Hilbert space $\mathcal{H}$ which satisfy the following properties:

1. $\mathcal{M}$ is isomorphic to $\mathcal{M}_0$,
2. there exists a cyclic and separating unit vector u for $\mathcal{M}$,
3. Δ_0 is the modular operator of $(\mathcal{M}, u)$.

The set of all von **N**eumann **A**lgebras satisfying 1-3 is denoted by

$$NA(\Delta_0; \mathcal{M}_0).$$

In [Wol92, Wol97, Wol98] the above problems were only considered in the factor case. Since it involves the same methods we consider here also the case with non-trivial center.

In addition to the given motivation, the above problems occur naturally in Tomita-Takesaki modular theory. Moreover, Borchers [Bor93] and Schroer [Sch97] posed similar problems in the framework of algebraic quantum field theory. Furthermore, these problems are closely related to the characterization of modular operators for a fixed von Neumann algebra unitarily equivalent to a given one and to the question of conjugacy of the modular automorphism groups of a given algebra (see §5.2 and §6.1).

5.2 General Remarks on the Inverse Problems

This section is concerned with a first analysis of the inverse problems formulated in §5.1. We establish relations between the different problems and show how the solutions of all inverse problems can be computed if one knows the set $NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ (the solutions of Problem 5.1.2). This legitimates the restriction to Problem 5.1.2 in the remaining parts of this thesis. Moreover, we introduce two classes of simple solutions of the inverse problems and prove useful reformulations of them. Furthermore, we cite a general theorem for Problem 5.1.4. Finally, we show how a more general inverse problem can sometimes be reduced to Problem 5.1.2.

Throughout this section, (Δ_0, J_0) are the modular objects for the pair $(\mathcal{M}_0, u_0)$ where $u_0 \in \mathcal{H}$ is a cyclic and separating unit vector for the von Neumann algebra $\mathcal{M}_0$. Furthermore, all cyclic and separating vectors of this section are assumed to be unit vectors, i. e. $\|u\| = 1$ for all cyclic and separating vectors u.

Notation. The following sets will prove useful in the investigation of the inverse problems.

(i) Let $\mathcal{W}_1$ be the set of all unitaries W on $\mathcal{H}$ such that $\mathrm{W}^* u_0 = \mathrm{C} u_0$ for a selfadjoint invertible operator $\mathrm{C} \eta \mathcal{Z}(\mathcal{M})$ and W commutes with Δ_0 and J_0.

(ii) Let $\mathcal{W}_2$ be the set of all unitaries W on $\mathcal{H}$ such that $\mathrm{W}^* u_0 = \mathrm{C} u_0$ for a selfadjoint invertible operator $\mathrm{C} \eta \mathcal{Z}(\mathcal{M})$ and W commutes with Δ_0.

(iii) Let $\mathcal{W}_3$ be the set of all unitaries W on $\mathcal{H}$ commuting with Δ_0 and J_0.

(iv) Let $\mathcal{W}_4$ be the set of all unitaries W on $\mathcal{H}$ commuting with Δ_0.

Proposition 5.2.1. *The following relations hold:*

$$\begin{aligned} NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) &= \mathrm{W}(NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0))\mathrm{W}^* \quad (\mathrm{W} \in \mathcal{W}_1), \\ NA(\Delta_0, u_0; \mathcal{M}_0) &= \bigcup_{\mathrm{W} \in \mathcal{W}_2} \mathrm{W}(NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0))\mathrm{W}^*, \\ NA(\Delta_0, \mathrm{J}_0; \mathcal{M}_0) &= \bigcup_{\mathrm{W} \in \mathcal{W}_3} \mathrm{W}(NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0))\mathrm{W}^*, \\ NA(\Delta_0; \mathcal{M}_0) &= \bigcup_{\mathrm{W} \in \mathcal{W}_4} \mathrm{W}(NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0))\mathrm{W}^*. \end{aligned}$$

Proof. We only prove the last equality. The others follow in the same manner if we restrict the argument to the smaller subsets $\mathcal{W}_i$ $(i = 1, 2, 3)$.

1. Let $\mathcal{N} \in NA(\Delta_0; \mathcal{M}_0)$, i.e. there is a cyclic and separating vector u for $\mathcal{N}$ such that $(\mathcal{N}, u)$ has modular objects (Δ_0, J). Since J and J_0 belong to the same modular operator, Proposition 3.1 and 3.2 of [LMW00] imply that J has the form $\mathrm{J} = \mathrm{W}\mathrm{J}_0\mathrm{W}^*$ where $\mathrm{W} \in \mathcal{W}_4$ with $\mathrm{W}u_0 = u$. Note that in [LMW00] only the factor case was proved, however, the proof of the more general case used here is the same. Furthermore, $(\mathrm{W}^*\mathcal{N}\mathrm{W}, u_0)$ has modular objects (Δ_0, J_0) (see Corollary 2.1.5). This implies $\mathrm{W}^*\mathcal{N}\mathrm{W} \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ and $\mathcal{N} \in \mathrm{W}(NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0))\mathrm{W}^*$. We have thus proved
$$NA(\Delta_0; \mathcal{M}_0) \subseteq \bigcup_{\mathrm{W} \in \mathcal{W}_4} \mathrm{W}(NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0))\mathrm{W}^*.$$

2. The other inclusion is a consequence of the transformation property of modular objects under unitarily implemented automorphisms (see Corollary 2.1.5). ☐

Proposition 5.2.1 leads immediately to the following definitions:

$$\begin{aligned} NA^1(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) &:= \{\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* | \mathrm{U} \in \mathcal{W}_1\}, \\ NA^1(\Delta_0, u_0; \mathcal{M}_0) &:= \{\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* | \mathrm{U} \in \mathcal{W}_2\}, \\ NA^1(\Delta_0, \mathrm{J}_0; \mathcal{M}_0) &:= \{\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* | \mathrm{U} \in \mathcal{W}_3\}, \\ NA^1(\Delta_0; \mathcal{M}_0) &:= \{\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* | \mathrm{U} \in \mathcal{W}_4\}. \end{aligned} \tag{5.2.1}$$

Corollary 5.2.2. *The following inclusions hold*

$$\begin{aligned} NA^1(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) &\subset NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0), \\ NA^1(\Delta_0, u_0; \mathcal{M}_0) &\subset NA(\Delta_0, u_0; \mathcal{M}_0), \\ NA^1(\Delta_0, \mathrm{J}_0; \mathcal{M}_0) &\subset NA(\Delta_0, \mathrm{J}_0; \mathcal{M}_0), \\ NA^1(\Delta_0; \mathcal{M}_0) &\subset NA(\Delta_0; \mathcal{M}_0). \end{aligned}$$

Proof. The inclusions follow directly from Proposition 5.2.1. ☐

The following proposition implies that the investigation of von Neumann algebras in $NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ can be transformed into the investigation of modular objects of $\mathcal{M}_0$.

Proposition 5.2.3. *An algebra $\mathcal{M}$ belongs to $NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ if and only if there exists a unitary* U *such that*

(i) $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$,

(ii) $u := \mathrm{U}^* u_0$ *is a cyclic and separating unit vector for* $\mathcal{M}_0$,

(iii) $(\mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ *are the modular objects of* $(\mathcal{M}_0, u)$, *and*

(iv) U *commutes with* J_0.

Proof. 1. Suppose $\mathcal{M} \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$. The unitary implementation theorem (Theorem 2.1.10) implies the existence of a unitary U such that $\mathcal{M} = \pi(\mathcal{M}_0) = \mathrm{ad}\,\mathrm{U}(\mathcal{M}_0)$ and $[\mathrm{U}, \mathrm{J}_0] = 0$. Since (Δ_0, J_0) are the modular objects of $(\mathcal{M}, u_0)$ we deduce from Corollary 2.1.5 that $\mathrm{U}^* u_0 = u$ is a cyclic and separating vector for $\mathcal{M}_0 = \mathrm{U}^*\mathcal{M}\mathrm{U}$ and that $\mathrm{U}^*\Delta_0\mathrm{U}$ and $\mathrm{U}^*\mathrm{J}_0\mathrm{U} = \mathrm{J}_0$ are the modular objects of $(\mathcal{M}_0, u)$.

2. Conversely, suppose that there is a unitary U satisfying (i) - (iv). Since $(\mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ are the modular objects for $(\mathcal{M}_0, \mathrm{U}^* u_0)$ it follows that u_0 is a cyclic and separating vector for $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$ and (Δ_0, J_0) are the modular objects for $(\mathcal{M}, u_0)$. □

Similar reasoning also yields analogous results for the other inverse problems. For instance, we present here the characterization for Problem 5.1.5:

Proposition 5.2.4. *An algebra $\mathcal{M}$ belongs to $NA(\Delta_0, \mathrm{J}_0; \mathcal{M}_0)$ if and only if there exists a unitary* U *such that*

(i) $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$,

(ii) $(\mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ *are the modular objects for* $(\mathcal{M}_0, u)$ *where* $u \in \mathcal{H}$ *is a cyclic and separating unit vector for* $\mathcal{M}_0$,

(iii) U *commutes with* J_0.

Remark 5.2.5. Corresponding propositions for Problem 5.1.4 and Problem 5.1.6 are obtained by omitting the commutativity of U and J_0.

The following example can be found in [Wol97] and [BW01] for the factor case.

Example 5.2.6. Suppose that $\Delta_0 = \mathrm{I}$ is the modular operator corresponding to u_0. Then the state $\langle \cdot u_0 | u_0 \rangle$ is a trace, u_0 is a trace vector, and $\mathcal{M}_0$ is finite.

Let now $\mathcal{M} \in NA(\mathrm{I}, \mathrm{J}_0, u_0; \mathcal{M}_0)$. Proposition 5.2.3 implies the existence of a unitary operator U such that $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$, the vector $\mathrm{U}^* u_0$ is cyclic and separating for $\mathcal{M}_0$, the modular objects of $(\mathcal{M}_0, \mathrm{U}^* u_0)$ are $(\mathrm{I}, \mathrm{J}_0)$, and J_0 commutes with U. On the other hand, we conclude from Lemma A.2.1 that there exists a selfadjoint invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M})$ such that $u_0 \in \mathcal{D}(\mathrm{C})$ and $\mathrm{U}^* u_0 = \mathrm{C} u_0$. Hence, $\mathrm{U} \in \mathcal{W}_1$ and $\mathcal{M}$ is also contained in $NA^1(\mathrm{I}, \mathrm{J}_0, u_0; \mathcal{M}_0)$. This implies

$$NA(\mathrm{I}, \mathrm{J}_0, u_0; \mathcal{M}_0) = NA^1(\mathrm{I}, \mathrm{J}_0, u_0; \mathcal{M}_0).$$

In the case $\Delta_0 = \mathrm{I}$ the inverse problem for the modular objects thus has a very simple solution.

Similarly, we can find $NA(\mathrm{I}, u_0; \mathcal{M}_0) = NA^1(\mathrm{I}, u_0; \mathcal{M}_0)$ and so on.

The following lemma states that the unitaries satisfying the properties (ii)-(iv) of Proposition 5.2.3 can be obtained by a transformation of the unitaries satisfying the properties (ii) and (iii) of Proposition 5.2.4. This sometimes simplifies the problem to find elements in $NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$.

Lemma 5.2.7. *Suppose* U *is a unitary and* u *is a cyclic and separating unit vector for* $\mathcal{M}_0$ *such that* $(\mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ *are the modular objects for* $(\mathcal{M}_0, u)$ *and* U *commutes with* J_0. *Then there exists a unitary* U_0 *such that*

$$\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{U}_0^*\Delta_0\mathrm{U}_0, \quad \mathrm{J}_0 = \mathrm{U}_0^*\mathrm{J}_0\mathrm{U}_0, \quad \textit{and} \quad \mathrm{U}_0^* u_0 = u.$$

Proof. Since $(\mathcal{M}_0, u)$ has modular objects $(\Delta := \mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ we have $\Delta u = u$ and $\mathrm{J}_0 u = u$. Setting $u_1 := \mathrm{U}u$ we get

$$\Delta_0 u_1 = \mathrm{U}\Delta\mathrm{U}^*\mathrm{U}u = \mathrm{U}\Delta u = \mathrm{U}u = u_1$$

and

$$\mathrm{J}_0 u_1 = \mathrm{J}_0\mathrm{U}u = \mathrm{U}\mathrm{J}_0 u = \mathrm{U}u = u_1.$$

Applying Lemma 5.2 of [LMW00] we now obtain the existence of a unitary operator V_1 such that

$$\mathrm{V}_1 u_0 = u_1, \quad \mathrm{V}_1\Delta_0 = \Delta_0\mathrm{V}_1, \quad \text{and} \quad \mathrm{V}_1\mathrm{J}_0 = \mathrm{J}_0\mathrm{V}_1.$$

Setting $\mathrm{U}_0 := \mathrm{V}_1^*\mathrm{U}$ we get a unitary with the claimed properties. ☐

Remark 5.2.8.

1. The analogous results to Lemma 5.2.7 for Problem 5.1.4 and Problem 5.1.6 are obtained by omitting the commutativity of U and J_0.

2. Note that Proposition 5.2.3 and Lemma 5.2.7 imply the following fact: The set of all modular objects (Δ, J) for $\mathcal{M}_0$ such that Δ is unitarily equivalent to Δ_0 and $\mathrm{J} = \mathrm{J}_0$ is given by the unitaries which generate the set $NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ from $\mathcal{M}_0$. So the *inverse problem* for the modular objects (Δ_0, J_0) is closely related to the *direct problem* to determine all other modular operators of $\mathcal{M}_0$ which are unitarily equivalent to the given ones and have the same fixed modular conjugation J_0.

   Similarly, one can connect the *inverse problem* for the modular operator Δ_0 with the *direct problem* to determine all other modular operators for $\mathcal{M}_0$ which are unitarily equivalent to Δ_0.

3. Let $NA(\mathcal{M}_0)$ (respectively $NA(u_0, \mathcal{M}_0)$) denote the set of von Neumann algebras on $\mathcal{H}$ which are isomorphic to $\mathcal{M}_0$ (and have cyclic and separating vector u_0). Then

$$\begin{array}{ccccc}
\{\mathcal{M}_0\} \subset NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) & \subset & NA(\Delta_0, u_0; \mathcal{M}_0) & \subset & NA(u_0, \mathcal{M}_0) \\
\cap & & \cap & & \cap \\
\{\mathcal{M}_0\} \subset NA(\Delta_0, \mathrm{J}_0; \mathcal{M}_0) & \subset & NA(\Delta_0; \mathcal{M}_0) & \subset & NA(\mathcal{M}_0)
\end{array}$$

   where equality does not hold in general.

4. If $\mathcal{M} \in NA(\Delta_0, u_0; \mathcal{M}_0)$ and $\mathcal{M} \cap \mathcal{M}_0$ has cyclic vector u_0 the algebras $\mathcal{M}_0$ and $\mathcal{M}$ coincide (this follows from [KR86, 9.2.36] and the KMS-condition). Thus, a non-trivial solution $\mathcal{M}$ of the inverse problem for the modular operator has to differ "significantly" from $\mathcal{M}_0$. In particular, it is not possible that $\mathcal{M}$ is a proper subalgebra of $\mathcal{M}_0$.

Let us now cite a theorem describing the solutions of the inverse problem of the modular operator of $(\mathcal{M}_0, u_0)$ (Problem 5.1.4). Corollary 5.2.2 implies that a von Neumann algebra $\mathcal{M} := \mathrm{U}\mathcal{M}_0\mathrm{U}^*$ belongs to $NA(\Delta_0, u_0; \mathcal{M}_0)$ if U satisfies

$$\mathrm{U} = \mathrm{KV} \quad \text{for } \mathrm{K} \in \mathcal{W}_2 \quad \text{and} \quad \operatorname{ad} V \in \operatorname{aut}(\mathcal{M}_0). \tag{5.2.2}$$

The example $\Delta_0 = \mathrm{I}$ demonstrates that this sufficient condition is sometimes also a necessary condition. This implies, at least for special Δ_0 and $\mathcal{M}_0$, the relation $NA(\Delta_0, u_0; \mathcal{M}_0) = NA^1(\Delta_0, u_0; \mathcal{M}_0)$. We will later see that for type I_∞ algebras where Δ_0 has a "generic spectrum" this relation holds as well (see §6.5). On the other hand, we will see that there exist examples of modular operators Δ_0 for type I_∞ algebras $\mathcal{M}_0$ (and other algebras) such that not all solutions $\mathcal{M} := \mathrm{U}\mathcal{M}_0\mathrm{U}^*$ of the inverse problem for the modular operator can be described by unitaries U satisfying the simple relations (5.2.2) (see §6.2 and §6.3).

The following theorem, which was first proved by Wollenberg [Wol97], gives a description of all von Neumann factors $\mathcal{M} := \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in NA(\Delta_0, u_0; \mathcal{M}_0)$ in terms of the unitaries U which is similar to the one given by (5.2.2). To get such a description we have to extend the Hilbert space $\mathcal{H}$ on which the von Neumann factors $\mathcal{M}_0$ and $\mathcal{M}$ act (a known trick, see e.g. [Con73] or [Str81, § 5]).

Let $\mathcal{F}_\infty = L(L_2(\mathbb{R}, \lambda))$ and ρ be the n.s.f. weight on $\mathcal{F}_\infty$ whose modular automorphism group σ_ρ^t is the group of all real translations on $\mathcal{F}_\infty$ (cf. e.g. [Str81, 4.15] for the existence). Furthermore, let $\mathcal{H}_\rho$ denote the GNS-space for $(\mathcal{F}_\infty, \rho)$. The isomorphism from $\mathcal{F}_\infty$ onto its GNS-representation is denoted by π_ρ and we set $\mathcal{F}_\rho := \pi_\rho(\mathcal{F}_\infty)$. The modular objects for $(\mathcal{F}_\rho, \rho)$ are denoted by $(\Delta_\rho, \mathrm{J}_\rho)$. Moreover, we use the following notations:

- $\rho_1(\cdot) := \rho \circ \pi_\rho^{-1}(\cdot)$ (a weight on $\mathcal{F}_\rho$),
- $\omega_0(\cdot) := \langle \cdot u_0 | u_0 \rangle$, $\omega_\mathrm{U}(\cdot) := \langle \cdot \mathrm{U}^* u_0 | \mathrm{U}^* u_0 \rangle$ (states on $\mathcal{M}_0$ and $\mathcal{M}_0'$),
- $\tilde{\sigma}_t := \operatorname{ad}(\Delta_0^{it} \otimes \Delta_\rho^{it})$ for $t \in \mathbb{R}$ (automorphisms on $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ and $\mathcal{M}_0' \otimes \mathcal{F}_\rho'$).

Theorem 5.2.9. *A von Neumann factor $\mathcal{M}$ belongs to $NA(\Delta_0, u_0; \mathcal{M}_0)$ if and only if there exists a unitary operator* U *such that*

(i) $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$,

(ii) there are unitaries $\mathrm{K}, \mathrm{Y}_1, \mathrm{Y}_2$ *on* $\mathcal{H} \otimes \mathcal{H}_\rho$ *such that* $\mathrm{K} \in \{\Delta_0 \otimes \Delta_\rho\}'$ *and*

$$\mathrm{U} \otimes \mathrm{I}_\rho = \mathrm{K} \cdot \mathrm{Y}_1 \cdot \mathrm{Y}_2, \quad \mathrm{Y}_1 \in \mathcal{M}_0 \otimes \mathcal{F}_\rho, \quad \mathrm{Y}_2 \in \mathcal{M}_0' \otimes \mathcal{F}_\rho', \tag{5.2.3}$$

(iii) $\omega_0 \otimes \rho_1(\cdot) = c(\omega_0 \otimes \rho_1)(\mathrm{K} \cdot \mathrm{K}^*)$ *as weights on* $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ *and* $\mathcal{M}_0' \otimes \mathcal{F}_\rho$ *where* $c > 0$ *and* $(\omega_0 \otimes \rho_1)(\mathrm{K} \cdot \mathrm{K}^*) := (\omega_\mathrm{U} \otimes \rho_1)(\mathrm{Y}_2^* \mathrm{Y}_1^* \cdot \mathrm{Y}_1 \mathrm{Y}_2)$.

For a proof of this theorem we refer to [BW01] or Appendix B.

In addition to the inverse problems presented in §5.1, it is also possible to investigate other, more general, inverse problems related to modular theory. For instance, one can look for non-isomorphic algebras acting on the same Hilbert space which have the same modular objects corresponding to the same cyclic and separating vector. In the type I case this is equivalent to Problem 5.1.2 since all type I_n algebras (for a fixed $n \in \mathbb{N} \cup \{\infty\}$) with isomorphic centers are isomorphic (see Theorem 4.2.1).

The other cases can often be reduced to Problem 5.1.2 as well:

Proposition 5.2.10. *Let $\mathcal{M}_i$, $i = 1, 2$, be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector $u_i \in \mathcal{H}$ and corresponding modular objects (Δ_i, J_i). Suppose that $\Delta_2 = \mathrm{U}\Delta_1\mathrm{U}^*$ where $\mathrm{U} \in \mathcal{U}(\mathcal{H})$. Then there exists a unitary $\mathrm{W} \in \mathcal{U}(\mathcal{H})$ such that u_1 is a cyclic and separating vector for $\mathrm{W}\mathcal{M}_2\mathrm{W}^*$ with modular objects (Δ_1, J_1).*

Proof. Note first that $(\mathrm{U}^*\mathcal{M}_2\mathrm{U}, \mathrm{U}^*u_2)$ has modular objects $(\Delta_1, \mathrm{U}^*\mathrm{J}_2\mathrm{U})$ (see Corollary 2.1.5). Then [LMW00, Proposition 3.1] implies that there is a unitary $\mathrm{W}_1 \in \mathcal{U}(\mathcal{H})$ commuting with Δ_1 such that $\mathrm{J}_1 = \mathrm{W}_1\mathrm{U}^*\mathrm{J}_2\mathrm{U}\mathrm{W}_1^*$. In addition, $u := \mathrm{W}_1\mathrm{U}^*u_2$ is a cyclic and separating vector for $\mathcal{N} := \mathrm{W}_1\mathrm{U}^*\mathcal{M}_2\mathrm{U}\mathrm{W}_1^*$ with modular objects (Δ_1, J_1). According to [LMW00, Proposition 3.2] there exists a unitary $\mathrm{W}_2 \in \mathcal{U}(\mathcal{H})$ commuting with Δ_1 and J_1 such that $u_1 = \mathrm{W}_2 u$ is a cyclic and separating vector for $\mathrm{W}_2\mathcal{N}\mathrm{W}_2^*$ with modular objects (Δ_1, J_1). Setting $\mathrm{W} := \mathrm{W}_2\mathrm{W}_1\mathrm{U}^*$ we get the assertion. □

Let now $\mathcal{M}_0$ be a type II_n von Neumann algebra ($n \in \{1, \infty\}$) with cyclic and separating vector u_0 and corresponding modular objects (Δ_0, J_0). Assume that u_0 is diagonalizable with respect to the center. Then the proof of Theorem 4.3.12 implies that every type II_n von Neumann algebra $\mathcal{M}$ with center isomorphic to the center $\mathcal{M}_0$ has a cyclic and separating vector whose modular operator is unitarily equivalent to Δ_0. According to Proposition 5.2.10 there is an algebra $\mathcal{N}$ in the isomorphism class of $\mathcal{M}$ which is a solution of the above mentioned more general inverse problem. The other solutions in the same isomorphism class are again solutions of Problem 5.1.2 with respect to $(\mathcal{N}, u_0)$.

5.3 Equivalence Relations in the Set of Solutions

From now on we consider only the inverse problem for the modular objects of $(\mathcal{M}_0, u_0)$ (Problem 5.1.2). We call this "the inverse problem" for short. The other inverse problems can be treated similarly (cf. also Proposition 5.3.5).

To classify the solutions of the inverse problem we introduce an equivalence relation using the simple class of solutions NA^1 (see (5.2.1)). The classification of all solutions will be formulated in terms of this equivalence relation in Chapter 6.

Definition 5.3.1. Two von Neumann algebras $\mathcal{M},\mathcal{N} \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ are called *equivalent solutions of the inverse problem*,

$$\mathcal{M} \sim \mathcal{N},$$

if $\mathcal{M} \in NA^1(\Delta_0, \mathrm{J}_0, u_0; \mathcal{N})$.

Proposition 5.3.2. *The relation "∼" is an equivalence relation.*

Proof. The relation "∼" is reflexive (choose $\mathrm{U} = \mathrm{I}$) and symmetric (replace U by U^*). To show that it is transitive, let $\mathcal{M} \sim \mathcal{N}$ and $\mathcal{N} \sim \mathcal{R}$, i. e. $\mathcal{M} = \mathrm{U}\mathcal{N}\mathrm{U}^*$ and $\mathcal{N} = \mathrm{V}\mathcal{R}\mathrm{V}^*$ where U and V are unitaries commuting with Δ_0 and J_0 such that $\mathrm{U}^* u_0 = \mathrm{C}_1 u_0$ $(\mathrm{C}_1 \eta \mathcal{N})$ and $\mathrm{V}^* u_0 = \mathrm{C}_2 u_0$ $(\mathrm{C}_2 \eta \mathcal{R})$. Setting $\tilde{\mathrm{U}} := \mathrm{UV}$ we get a unitary $\tilde{\mathrm{U}}$ such that $\mathcal{M} = \tilde{\mathrm{U}}\mathcal{R}\tilde{\mathrm{U}}^*$ and $\tilde{\mathrm{U}}$ commutes with Δ_0 and J_0. Furthermore, with $\tilde{\mathrm{C}}_1 := \mathrm{V}^*\mathrm{C}_1\mathrm{V} \eta \mathcal{R}$ we have

$$\tilde{\mathrm{U}}^* u_0 = \mathrm{V}^*\mathrm{U}^* u_0 = \mathrm{V}^*\mathrm{C}_1 u_0 = \tilde{\mathrm{C}}_1 \mathrm{V}^* u_0 = \tilde{\mathrm{C}}_1 \mathrm{C}_2 u_0$$

where $\tilde{\mathrm{C}}_1\mathrm{C}_2 \eta \mathcal{R}$. Thus $\mathcal{M} \sim \mathcal{R}$. □

Remark 5.3.3.

1. Similar equivalence relations can be introduced also for the other inverse problems if we substitute $\mathcal{W}_1$ by $\mathcal{W}_i$ $(i = 2, 3, 4)$.
2. The definition of the equivalence relation obviously implies that the simple class NA^1 defined by (5.2.1) is an equivalence class with respect to $\sim$.

In the following we consider a second simple class of solutions for the inverse problem. This class was introduced in [Wol97] and [BW01] for type I factors. For this purpose let (Δ_0, J_0) be the modular objects of $(\mathcal{M}_0, u_0)$ and assume for the moment that there is a cyclic and separating vector $u_1 \in \mathcal{H}$ such that $(\Delta_0^{-1}, \mathrm{J}_0)$ are the modular objects of $(\mathcal{M}_0, u_1)$. The existence of such a vector will proved unter certain conditions later (Lemma 5.3.7). Lemma A.3.1 then implies the existence of a conjugation L such that L commutes with Δ_0 and J_0, and $\mathrm{LC}u_i = \mathrm{C}^* u_i$, $i = 0, 1$, for all $\mathrm{C} \in \mathcal{Z}(\mathcal{M}_0)$. Setting $\mathrm{U}_1 := \mathrm{LJ}_0$ we get a unitary U_1 such that U_1 commutes with J_0, $\mathrm{U}_1\mathrm{C}u_i = \mathrm{C}u_i$ for all $\mathrm{C} \in \mathcal{Z}(\mathcal{M}_0)$, and

$$\mathrm{U}_1^* \Delta_0 \mathrm{U}_1 = \Delta_0^{-1}.$$

We can now define the following class of von Neumann algebras solving the inverse problem:

$$\begin{aligned} NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) := \{&\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* | \mathrm{U} = \mathrm{KU}_1, \mathrm{K} \in \mathcal{W}_3, \\ &\mathrm{K}^* u_0 = \mathrm{C}u_1 \text{ with a selfadjoint invertible } \mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)\}. \end{aligned} \tag{5.3.1}$$

Lemma 5.3.4. *Under the above assumptions the definition of NA^2 is independent of the pair* (U_1, u_1) *and*

$$\emptyset \neq NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) \subset NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0).$$

The proof is the same as for type I factors (cf. [BW01]). Nevertheless, we repeat it for the reader's convenience with some additional remarks:

Proof. 1. We first prove the existence of a unitaries $\mathrm{K} \in \mathcal{W}_3$ such that $\mathrm{K}^* u_0 = \mathrm{C} u_1$ for a selfadjoint invertible $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$. Since u_0 and u_1 are cyclic and separating vectors with modular objects (Δ_0, J_0) and $(\Delta_0^{-1}, \mathrm{J}_0)$, respectively, we have $\Delta_0 u_i = u_i$ and $\mathrm{J}_0 u_i = u_i$ for $i = 0, 1$. Lemma 5.2 in [LMW00] now yields the existence of a unitary K commuting with Δ_0 and J_0 such that $\mathrm{K}^* u_0 = u_1$.

2. To verify the independence of the definition from the pair (U_1, u_1), let (U_2, u_2) be another pair which fulfils the above conditions in place of (U_1, u_1). Let further $\mathrm{K}_2 \in \mathcal{W}_3$ such that $\mathrm{K}_2^* u_0 = \mathrm{C}_2 u_2$ where $\mathrm{C}_2\eta\mathcal{Z}(\mathcal{M}_0)$ is selfadjoint and invertible. Since u_1 and u_2 have the same modular objects Lemma A.2.1 implies the existence of an invertible and selfadjoint operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ such that $u_2 = \mathrm{C}u_1$. Setting $\mathrm{K}_1 := \mathrm{K}_2\mathrm{U}_2\mathrm{U}_1^*$ we get a unitary commuting with Δ_0 and J_0 such that $\mathrm{K}_1\mathrm{U}_1 = \mathrm{K}_2\mathrm{U}_2$ and
$$\mathrm{K}_1^* u_0 = \mathrm{U}_1\mathrm{U}_2^*\mathrm{K}_2^* u_0 = \mathrm{U}_1\mathrm{U}_2^*\mathrm{C}_2 u_2 = \mathrm{U}_1\mathrm{C}_2 u_2 = \mathrm{U}_1\mathrm{C}_2\mathrm{C}u_1 = \mathrm{C}_2\mathrm{C}u_1.$$
Therefore, we have
$$\mathrm{K}_2\mathrm{U}_2\mathcal{M}_0\mathrm{U}_2^*\mathrm{K}_2^* = \mathrm{K}_1\mathrm{U}_1\mathcal{M}_0\mathrm{U}_1^*\mathrm{K}_1^* \in NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0).$$

3. Let now $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* = \mathrm{K}\mathrm{U}_1\mathcal{M}\mathrm{U}_1^*\mathrm{K}^* \in NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$. We show that $\mathcal{M} \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$. In fact, since $(\Delta_0^{-1}, \mathrm{J}_0)$ are the modular objects for $(\mathcal{M}_0, \mathrm{C}u_1)$ and
$$\begin{aligned} \mathrm{U}^*\Delta_0\mathrm{U} &= \mathrm{U}_1^*\mathrm{K}^*\Delta_0\mathrm{K}\mathrm{U}_1 = \mathrm{U}_1^*\Delta_0\mathrm{U}_1 = \Delta_0^{-1}, \\ \mathrm{U}^*\mathrm{J}_0\mathrm{U} &= \mathrm{U}_1^*\mathrm{K}^*\mathrm{J}_0\mathrm{K}\mathrm{U}_1 = \mathrm{U}_1^*\mathrm{J}_0\mathrm{U}_1 = \mathrm{J}_0, \\ \mathrm{U}^* u_0 &= \mathrm{U}_1^*\mathrm{K}^* u_0 = \mathrm{U}_1^*\mathrm{C}u_1 = \mathrm{C}u_1 \end{aligned}$$
Proposition 5.2.3 implies $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$. ☐

Proposition 5.3.5. *The class $NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ is an equivalence class with respect to "$\sim$".*

Proof. Let $\mathcal{M}_1, \mathcal{M}_2 \in NA(\Delta_0, \mathrm{J}_0, U_0; \mathcal{M}_0)$ be two solutions of the inverse problem.

1. Suppose that $\mathcal{M}_i \in NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$, $i = 1, 2$. Then there exist unitaries $\mathrm{K}_i \in \mathcal{W}_3$ such that $\mathrm{K}_i^* u_0 = \mathrm{C}_i u_1$ for selfadjoint invertible operators $\mathrm{C}_i\eta\mathcal{Z}(\mathcal{M}_0)$, and $\mathcal{M}_i = \mathrm{K}_i\mathrm{U}_1\mathcal{M}_0\mathrm{U}_1^*\mathrm{K}_i^*$. Defining
$$\mathrm{W} := \mathrm{K}_1\mathrm{U}_1\mathrm{U}_1^*\mathrm{K}_2^* = \mathrm{K}_1\mathrm{K}_2^*$$
we have $\mathrm{W}\mathcal{M}_2\mathrm{W}^* = \mathrm{K}_1\mathrm{U}_1\mathcal{M}_0\mathrm{U}_1^*\mathrm{K}_1^* = \mathcal{M}_1$ and
$$\begin{aligned} \mathrm{W}\Delta_0 &= \mathrm{K}_1\mathrm{K}_2^*\Delta_0 = \Delta_0\mathrm{K}_1\mathrm{K}_2^* = \Delta_0\mathrm{W} \\ \mathrm{W}\mathrm{J}_0 &= \mathrm{K}_1\mathrm{K}_2^*\mathrm{J}_0 = \mathrm{J}_0\mathrm{K}_1\mathrm{K}_2^* = \mathrm{J}_0\mathrm{W} \\ \mathrm{W}^* u_0 &= \mathrm{K}_2\mathrm{K}_1^* u_0 = \mathrm{K}_2\mathrm{C}_1 u_1 = \mathrm{K}_2\mathrm{U}_1\mathrm{C}_1 u_1 = \underbrace{\mathrm{K}_2\mathrm{U}_1\mathrm{C}_1\mathrm{C}_2^{-1}\mathrm{U}_1^*\mathrm{K}_2^*}_{=:\mathrm{C}} u_0. \end{aligned}$$
Note that we have used $\mathrm{C}_2 u_1 = \mathrm{U}_1^*\mathrm{C}_2 u_1 = \mathrm{U}_1^*\mathrm{K}_2^* u_0$ in the last step. Since $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_2)$ the conditions of Definition 5.3.1 are fulfilled and hence $\mathcal{M}_1 \sim \mathcal{M}_2$.

2. Let now $\mathcal{M}_1 \in NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$. This implies the existence of a unitary $\mathrm{K} \in \mathcal{W}_3$ such that $\mathrm{K}^* u_0 = \mathrm{C}_1 u_1$ with $\mathrm{C}_1 \eta \mathcal{Z}(\mathcal{M}_0)$, and $\mathcal{M}_1 = \mathrm{K}\mathrm{U}_1 \mathcal{M}_0 \mathrm{U}_1^* \mathrm{K}^*$. Suppose that $\mathcal{M}_2 \sim \mathcal{M}_1$. Then there exists a unitary $\mathrm{U} \in \mathcal{W}_1$ such that $\mathrm{U}^* u_0 = \mathrm{C}_2 u_0$ with $\mathrm{C}_2 \eta \mathcal{Z}(\mathcal{M}_1)$ and $\mathcal{M}_2 = \mathrm{U}\mathcal{M}_1\mathrm{U}^*$. Setting $\tilde{\mathrm{K}} := \mathrm{UK}$ we get a unitary commuting with Δ_0 and J_0 such that
$$\begin{aligned}\tilde{\mathrm{K}}^* u_0 &= \mathrm{K}^*\mathrm{U}^* u_0 = \mathrm{K}^*\mathrm{C}_2 u_0 = \mathrm{K}^*\mathrm{C}_2\mathrm{K}\mathrm{C}_1 u_1 \\ &= \mathrm{U}_1 \underbrace{\mathrm{U}_1^*\mathrm{K}^*\mathrm{C}_2\mathrm{K}\mathrm{U}_1\mathrm{C}_1}_{=:\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)} u_1 = \mathrm{C}u_1,\end{aligned}$$
and $\tilde{\mathrm{K}}\mathrm{U}_1\mathcal{M}_0\mathrm{U}_1^*\tilde{\mathrm{K}}^* = \mathcal{M}_2$. This implies $\mathcal{M}_2 \in NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$. □

In the following we investigate under which conditions Δ_0^{-1} can also be a modular operator corresponding to a cyclic and separating vector u_1. The next proposition suggested by Wollenberg states that this is only possible if the algebra $\mathcal{M}_0$ is semifinite.

Proposition 5.3.6. *Let $\mathcal{M}_0$ be a von Neumann algebra, Δ_0 a modular operator corresponding to an n. s. f. weight φ such that Δ_0^{-1} is also a modular operator corresponding to an n. s. f. weight. Then $\mathcal{M}_0$ is semifinite.*

Proof. Suppose that Δ^{-1} is the modular operator for some weight ψ. The Unitary Cocycle Theorem (Theorem 2.1.15) implies the existence of a unitary cocycle $\mathrm{U}_t = [D\varphi : D\psi]_t \in \mathcal{M}_0$ such that $\operatorname{ad}\Delta_0^{-it} = \operatorname{ad}(\mathrm{U}_t\Delta_0^{it})$ for $t \in \mathbb{R}$. Then $\mathrm{V}_t := \Delta_0^{-it}(\mathrm{U}_t^*\Delta_0^{-it}) \in \mathcal{M}_0'$ for all $t \in \mathbb{R}$. Now $\Delta_0^{-it} = (\mathrm{U}_t\Delta_0^{it})\mathrm{V}_t$ and $\Delta_0^{it} = (\Delta_0^{-it})^* = \mathrm{V}_t^*\mathrm{U}_{-t}\Delta_0^{-it}$ implies $\Delta_0^{2it} = \mathrm{V}_t^*\mathrm{U}_{-t}$. From the latter it follows that $\operatorname{ad}\Delta_0^{it}$ is an inner automorphism for all $t \in \mathbb{R}$ which is only possible if $\mathcal{M}_0$ is semifinite (cf. §2.3). □

Legitimated by Proposition 5.3.6 we can assume in the following investigations that $\mathcal{M}_0$ is a semifinite algebra and $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$ where $\mathrm{H}_0\eta\mathcal{M}_0$ with $\operatorname{tr}(\mathrm{H}_0) < \infty$ (cf. Theorem 3.3.9). This implies $\Delta_0^{-1} = \mathrm{H}_0^{-1}\mathrm{J}_0\mathrm{H}_0\mathrm{J}_0$.

Lemma 5.3.7. *Let $\mathcal{M}_0$ be a semifinite von Neumann algebra, and let Δ_0, J_0, H_0 be as above. Then the following statements are equivalent:*

(i) $(\Delta_0^{-1}, \mathrm{J}_0)$ *are the modular objects corresponding to a cyclic and separating vector $u_1 \in \mathcal{H}$ for $\mathcal{M}_0$.*

(ii) *There is a positive invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ such that*
$$\operatorname{tr}(\mathrm{C}\mathrm{H}_0^{-1}) < \infty. \tag{5.3.2}$$

Proof. *(i)* ⇒ *(ii)*: Suppose that $(\Delta_0^{-1}, \mathrm{J}_0)$ are the modular objects corresponding to a cyclic and separating vector u_1 for $\mathcal{M}_0$. The results of §3.2 and §3.3 now imply the existence of an invertible operator $\mathrm{T}_{u_1}\eta\mathcal{M}_0$ corresponding to u_1 such that $\operatorname{tr}(\mathrm{T}_{u_1}\mathrm{T}_{u_1}^*) = \operatorname{tr}(\mathrm{T}_{u_1}^*\mathrm{T}_{u_1}) < \infty$ and
$$\mathrm{H}_0^{-1}\mathrm{J}_0\mathrm{H}_0\mathrm{J}_0 = \Delta_0^{-1} = \mathrm{SS}^*\mathrm{J}_0(\mathrm{SS}^*)^{-1}\mathrm{J}_0.$$

Since the decomposition of Δ_0^{-1} into positive operators is unique up to a positive, invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ (see Lemma A.1.1) we conclude $\mathrm{CH}_0^{-1} = \mathrm{T}_{u_1}\mathrm{T}_{u_1}^*$ and

$$\mathrm{tr}(\mathrm{CH}_0^{-1}) = \mathrm{tr}(\mathrm{T}_{u_1}\mathrm{T}_{u_1}^*) < \infty.$$

(ii) $\Rightarrow$ *(i)*: Let $\mathrm{T}_{u_0} = \mathrm{H}_0^{1/2}\mathrm{V}$ be the polar decompostion of the operator T_{u_0} which corresponds to u_0 (see Corollary 3.2.4 and Corollary 3.3.6). Suppose that there is a positive invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ such that $\mathrm{tr}(\mathrm{CH}_0^{-1}) < \infty$. Setting $\mathrm{S} := \mathrm{C}^{1/2}\mathrm{H}_0^{-1/2}\mathrm{V}\eta\mathcal{M}_0$ we have

$$\mathrm{tr}(\mathrm{SS}^*) = \mathrm{tr}(\mathrm{CH}_0^{-1}) < \infty.$$

Hence, there exists a cyclic and separating vector u_1 corresponding to S such that

$$\mathrm{CH}_0^{-1}\mathrm{J}_0\mathrm{C}^{-1}\mathrm{H}_0\mathrm{J}_0 = \Delta_0^{-1}$$

and J_0 are the modular objects corresponding to u_1 (cf. Theorem 3.3.9). □

We now investigate under which conditions the second statement of Lemma 5.3.7 is true. For this purpose we must distinguish the different types:

Lemma 5.3.8. *(i) For type I_n algebras ($n \in \mathbb{N}$) condition* (5.3.2) *is always satisfied.*

(ii) For type I_∞ and type II_∞ algebras the condition (5.3.2) *is never satisfied.*

Proof. (i) Let $\mathcal{M}_0$ be a type I_n algebra ($n \in \mathbb{N}$). Then H_0 can be decomposed into $\mathrm{H}_0 = \sum_{k=1}^n f_k\mathrm{E}_k$ where $0 < f_k\eta\mathcal{Z}(\mathcal{M}_0)$ and $\mathrm{E}_k \in \mathcal{M}_0$ are pairwise orthogonal projections for $k = 1, \ldots, n$ (see Theorem 4.2.4). Moreover, we can assume that $\mathrm{C}_{\mathrm{E}_k} = \mathrm{supp}(f_k) = \mathrm{I}$ and $m_k = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k) = \mathrm{I}$. Note that this convention differs slightly from the convention used in §4.2. Defining $f := \inf_{k\in\{1,\ldots,n\}} f_k$ we get a positive invertible function affiliated with $\mathcal{Z}(\mathcal{M}_0)$. As in §4.2 we again assume without loss of generality that $\mathrm{tr} = \omega_\mu \circ \mathrm{tr}_{\mathcal{M}_0}$ where ω_μ is a weight on the center of $\mathcal{M}_0$ and $\mathrm{tr}_{\mathcal{M}_0}$ is the canonical central trace. Then

$$\mathrm{tr}(f\mathrm{H}_0^{-1}) = \mathrm{tr}(f\sum_{k=1}^n f_k^{-1}\mathrm{E}_k) = \sum_{k=1}^n \omega_\mu(ff_k^{-1}\,\mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k)) =$$
$$= \sum_{k=1}^n \omega_\mu(ff_k^{-1}) \le \sum_{k=1}^n \omega_\mu(f_kf_k^{-1}) \le n < \infty,$$

hence (5.3.2) is fulfilled if we substitute C by $f\eta\mathcal{Z}(\mathcal{M}_0)$.

(ii) (a) Let now $\mathcal{M}_0$ be a type I_∞ or II_∞ factor and $\mathrm{T}_{u_0} = \mathrm{H}_0^{1/2}\mathrm{V}\eta\mathcal{M}_0$ the operator corresponding to the cyclic and separating vector u_0.

Let further E be the spectral measure of H_0. We define a positive measure μ_{tr} on the σ-algebra of Borel sets in $\mathbb{R}$ by

$$\mu_{\mathrm{tr}}(B) := \mathrm{tr}(\mathrm{E}(B))$$

for all Borel sets $B \subset \mathbb{R}$. This measure fulfils

$$\int \lambda d\mu_{\mathrm{tr}}(\lambda) = \mathrm{tr}(\mathrm{H}_0) < \infty.$$

Without loss of generality we assume $\int \lambda d\mu_{\mathrm{tr}}(\lambda) = 1$. Then

$$1 = \int \lambda d\mu_{\mathrm{tr}}(\lambda) \geq \int\limits_{[0,1]} \lambda d\mu_{\mathrm{tr}}(\lambda) + \int\limits_{(1,\infty)} d\mu_{\mathrm{tr}}(\lambda).$$

In particular, one obtains

$$\int\limits_{(1,\infty)} d\mu_{\mathrm{tr}}(\lambda) < \infty.$$

Since $\mathcal{M}_0$ is infinite we have $\infty = \mathrm{tr}(\mathrm{I}) = \mu_{\mathrm{tr}}(\mathbb{R})$. This implies

$$\infty = \int\limits_{\mathbb{R}} d\mu_{\mathrm{tr}}(\lambda) = \int\limits_{[0,1]} d\mu_{\mathrm{tr}}(\lambda) + \underbrace{\int\limits_{(1,\infty)} d\mu_{\mathrm{tr}}(\lambda)}_{<\infty}$$

and

$$\int\limits_{[0,1]} d\mu_{\mathrm{tr}}(\lambda) = \infty.$$

Suppose now that also $\mathrm{tr}(\mathrm{H}_0^{-1}) < \infty$. Then

$$\infty > \int\limits_{\mathbb{R}} \lambda^{-1} d\mu_{\mathrm{tr}}(\lambda) \geq \underbrace{\int\limits_{[0,1]} d\mu_{\mathrm{tr}}(\lambda)}_{=\infty} + \int\limits_{(1,\infty)} \lambda^{-1} d\mu_{\mathrm{tr}}(\lambda)$$

is a contradiction.

(b) The non-factor case follows from the part (a) using the direct integral decomposition (cf. §4.2 and §4.3). □

Example 5.3.9. Let $\mathcal{R}(\mathcal{A}, \alpha)$ be the hyperfinite II_1 factor constructed in Example 2.2.6. Let further $\mathrm{M} = [\mathrm{U}(pq^{-1})\mathrm{A}(pq^{-1})]_{p,q \in G} \in \mathcal{R}$, where U is the unitary implementation of α on $\mathcal{H}$, and $\mathrm{A}(p) \in \mathcal{A} = L_\infty(S, \lambda)$ for all $p \in G$. The trace on $\mathcal{R}$ is the defined by

$$\mathrm{tr}(\mathrm{M}) = \int\limits_S \mathrm{A}(e) d\lambda.$$

Let Φ be the canonical homomorphism from $\mathcal{A}$ into $\mathcal{R}$, i.e. $\Phi(f) = [\delta_{p,q} f]_{p,q}$ for $f \in \mathcal{A}$. This implies

$$\operatorname{tr}(\Phi(f)^*\Phi(f)) < \infty \quad \Leftrightarrow \quad \int |f|^2 \, d\lambda < \infty.$$

We now define two functions $f_1, f_2 \in \mathcal{A}$ by $f_1 := x + 1$ and $f_2 := x$. Then

$$\int_0^1 (x+1)^2 d\lambda(x) < \infty \quad \text{and} \quad \int_0^1 (x+1)^{-2} d\lambda(x) < \infty$$

whereas

$$\int_0^1 x^2 d\lambda(x) < \infty \quad \text{and} \quad \int_0^1 x^{-2} d\lambda(x) = \infty.$$

This implies that condition (5.3.2) is satisfied for $\Phi(f_1)$ but not for $\Phi(f_2)$. Hence, the second simple class does not exist for all cyclic and separating vectors, but for some of them.

Since every type II_1 algebra has a hyperfinite II_1 subfactor [KR86, Exercice 12.4.25] this example also demonstrates this fact for all type II_1 factors. The non-factor case again follows with the help of the direct integral decomposition.

Remark 5.3.10. Example 5.3.9 implies that the spectral measure of $\Phi(f)$ is given by

$$\mathrm{E}_{\Phi(f)}(B) = [\chi_{f^{-1}(B)} \delta_{p,q}]_{p,q}$$

where χ_M is the characteristic function of the Borel set $M \subset \mathbb{R}$. Hence, $\Phi(f)$ has the same types of spectrum as f. Accordingly, the positive operator H_0 generating the modular operator can have all types of spectral points. This is a difference to the type I factor case. In the latter we have, also for I_∞ factors, only point spectrum (and eventually 0 as continuous spectrum) since H_0 is always a trace class operator.

Summarizing Proposition 5.3.6, Example 5.3.9, and Lemma 5.3.8 we have

Theorem 5.3.11. *The second simple class $NA^2(\Delta_0, \mathrm{J}_0, U_0; \mathcal{M}_0)$ of solutions of the inverse problem only exists for finite algebras $\mathcal{M}_0$. More precisely,*

1. *NA^2 exists for every cyclic and separating vector u_0 if $\mathcal{M}_0$ is a type I_n algebra ($n \in \mathbb{N}$).*

2. *NA^2 exists for some cyclic and separating vectors u_0, but not for all, if $\mathcal{M}_0$ is a type II_1 algebra.*

Chapter 6

Classification of the Solutions of the Inverse Problem

In this chapter we will use the equivalence relation introduced in §5.3 for the investigation of the inverse problem. We will first present some general results valid both for the type I and the type II case. The next two sections are concerned with the type I and the type II case, respectively. In both cases it is possible to completely classify the solutions with the help of some spectral invariants the construction of which is based on the results of Chapter 4 (Theorem 6.2.8 and Theorem 6.3.8). The fourth section contains some remarks on the type III_λ case $(0 < \lambda < 1)$. In the last section we will introduce a different equivalence relation for semifinite factors which is equivalent to the first one only in the type I case. Furthermore, we will exhibit a special class of modular operators, the so-called modular operators with generic spectrum, which have at most two classes of solutions with respect to this second equivalence relation (Theorem 6.5.12).

Most results of §6.1 are due to the author (see [Bol00a] and [Bol00b]), Corollary 6.1.4 and Corollary 6.1.5 were previously proved for type I factors by Wollenberg (see [Wol98], cf. also [BW01]). The results of §6.2 are essentially new. Some aspects of the inverse problem for type I factors were already considered by Wollenberg in [Wol98] (cf. also [BW01]). §6.3 and §6.4 are due to the author (see [Bol00a] and [Bol00b] for the factor case). The notion of generic spectrum (see Definition 6.5.10) was introduced by Wollenberg in [Wol98] (cf. also [BW01]) for type I factors whereas the generalization to semifinite factors is new.

As in the last section of the previous chapter we will focus on the inverse problem for the modular objects (Problem 5.1.2), the inverse problem for short. However, similar results can also be obtained for the other problems.

6.1 General Results for the Semifinite Case

Let now (Δ_0, J_0) be the modular objects of $(\mathcal{M}_0, u_0)$ for a semifinite algebra $\mathcal{M}_0$ and $\mathrm{T}_{u_0} = \mathrm{H}_0^{1/2}\mathrm{V}\eta\mathcal{M}_0$ be the invertible operator corresponding to the cyclic and separating vector $u_0 \in \mathcal{H}$. Then $\mathrm{tr}(\mathrm{H}_0) < \infty$ and $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$ (see

Theorem 3.3.9). Similarly to §4.2 and §4.3 we assume throughout this section that $\mathrm{tr} = \omega \circ \mathrm{tr}_{\mathcal{M}_0}$ where $\mathrm{tr}_{\mathcal{M}_0}$ is a central trace on $\mathcal{M}_0$ and ω is a normal faithful state on the center $\mathcal{Z}(\mathcal{M}_0)$ of $\mathcal{M}_0$. Furthermore, we can assume without loss of generality that

$$\mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_0) = \mathrm{I}. \tag{6.1.1}$$

Indeed, if $\mathrm{C} := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_0) \neq \mathrm{I}$ we replace u_0 by the vector $v_0 := \mathrm{C}^{-1/2} u_0$, where $u_0 \in \mathcal{D}(\mathrm{C}^{-1/2})$ because

$$\mathrm{tr}_{\mathrm{H}_0}(\mathrm{C}^{-1}) = \omega(\mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_0\mathrm{C}^{-1})) = \omega(\mathrm{C}\mathrm{C}^{-1}) = 1 < \infty.$$

v_0 is a cyclic and separating vector for $\mathcal{M}_0$ with modular objects (Δ_0, J_0) and corresponding operator $\mathrm{T}_{v_0} = \mathrm{C}^{-1/2}\mathrm{T}_{u_0} = \mathrm{C}^{-1/2}\mathrm{H}_0^{1/2}\mathrm{V}$ such that

$$\mathrm{tr}_{\mathcal{M}_0}(\mathrm{T}_{v_0}\mathrm{T}_{v_0}^*) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{C}^{-1}\mathrm{H}_0) = \mathrm{I}.$$

Moreover, by Lemma 5.2.7, there is a unitary W commuting with Δ_0 and J_0 such that $\mathrm{W}^* u_0 = v_0 = \mathrm{C}^{-1/2} u_0$. Hence, $\mathrm{W} \in \mathcal{W}_1$ and

$$NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) = \mathrm{W}(NA(\Delta_0, \mathrm{J}_0, v_0; \mathcal{M}_0))\mathrm{W}^* = NA(\Delta_0, \mathrm{J}_0, v_0; \mathcal{M}_0)$$

according to Proposition 5.2.1.

If an algebra $\mathcal{M}$ is a solution of the inverse problem for the modular objects of $(\mathcal{M}_0, u_0)$, then Proposition 5.2.3 implies the existence of a a unitary U such that $(\Delta := \mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ are the modular objects for $(\mathcal{M}_0, u := \mathrm{U}^* u_0)$. According to the results of §3.2 and §3.3 there is an invertible operator $\mathrm{T}_u \eta \mathcal{M}_0$ corresponding to u such that $\mathrm{tr}(\mathrm{T}_u\mathrm{T}_u^*) < \infty$. Moreover, setting $\mathrm{H} := \mathrm{T}_u\mathrm{T}_u^*$, Theorem 3.3.9 implies $\Delta = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$.

Assume now there is another unitary $\tilde{\mathrm{U}}$ commuting with J_0 such that $\tilde{\mathrm{U}}\mathcal{M}_0\tilde{\mathrm{U}}^* = \mathcal{M}$ and $\tilde{\mathrm{U}}^*\Delta_0\tilde{\mathrm{U}} = \tilde{\mathrm{H}}\mathrm{J}_0\tilde{\mathrm{H}}^{-1}\mathrm{J}_0$ with $\tilde{\mathrm{H}}\eta\mathcal{M}_0$. Defining $\mathrm{V} := \mathrm{U}^*\tilde{\mathrm{U}}$ we get a unitary such that $\mathrm{ad}\,\mathrm{V} \in \mathrm{aut}(\mathcal{M}_0)$ and

$$\mathrm{V}\tilde{\mathrm{H}}\mathrm{V}^*\mathrm{J}_0\mathrm{V}\tilde{\mathrm{H}}^{-1}\mathrm{V}^*\mathrm{J}_0 = \mathrm{U}^*\tilde{\mathrm{U}}\tilde{\mathrm{H}}\mathrm{J}_0\tilde{\mathrm{H}}^{-1}\mathrm{J}_0\tilde{\mathrm{U}}^*\mathrm{U} = \mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0.$$

By Lemma A.1.1, there is a positive invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ such that $\mathrm{V}\tilde{\mathrm{H}}\mathrm{V}^* = \mathrm{C}\mathrm{H}$. This proves that the positive invertible operator $\mathrm{H}\eta\mathcal{M}_0$ corresponding to a solution of the inverse problem is unique up to conjugacy.

We can even prove that the equivalence classes of the inverse problem are characterized by conjugacy classes of positive invertible operators:

Lemma 6.1.1. *Let $\mathcal{M}_1, \mathcal{M}_2 \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ be two solutions of the inverse problem, and let $\mathrm{H}_i\eta\mathcal{M}_0$ $(i = 1, 2)$ be the corresponding positive invertible operators. The following conditions are equivalent:*

(i) $\mathcal{M}_1 \sim \mathcal{M}_2$

(ii) There are a unitarily implemented automorphism $\alpha = \mathrm{ad}\,\mathrm{W}$, $\mathrm{W} \in \mathcal{U}(\mathcal{H})$, on $\mathcal{M}_0$ and a positive invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ such that $\alpha(\mathrm{H}_2) = \mathrm{C}\mathrm{H}_1$, i. e. $\mathrm{W}\mathrm{H}_2\mathrm{W}^* = \mathrm{C}\mathrm{H}_1$.

The operator W can be chosen in such a way that it commutes with J_0.

Proof. Since $\mathcal{M}_i$, $i = 1, 2$, is a solution of the inverse problem there is a unitary operator U_i on $\mathcal{H}$ commuting with J_0 such that $\mathcal{M}_i = \mathrm{U}_i \mathcal{M}_0 \mathrm{U}_i^*$ and

$$(\Delta_i := \mathrm{U}_i^* \Delta_0 \mathrm{U}_i =: \mathrm{H}_i \mathrm{J}_0 \mathrm{H}_i^{-1} \mathrm{J}_0, \mathrm{J}_0)$$

are the modular objects for $(\mathcal{M}_0, u_i := \mathrm{U}_i^* u_0)$.

1. Suppose that $\mathcal{M}_1 \sim \mathcal{M}_2$, i.e. there is a unitary V commuting with J_0 and Δ_0 such that $\mathcal{M}_1 = \mathrm{V}\mathcal{M}_2\mathrm{V}^*$. If we set $\mathrm{W} := \mathrm{U}_1^* \mathrm{V} \mathrm{U}_2$ a simple calculation yields $\alpha := \mathrm{ad}\,\mathrm{W} \in \mathrm{aut}(\mathcal{M}_0)$. Moreover, W commutes with J_0. Since

$$\begin{aligned}(\mathrm{W}\mathrm{H}_2\mathrm{W}^*)\mathrm{J}_0(\mathrm{W}\mathrm{H}_2^{-1}\mathrm{W}^*)\mathrm{J}_0 &= \mathrm{U}_1^*\mathrm{V}\Delta_0\mathrm{V}^*\mathrm{U}_1 \\ &= \mathrm{H}_1\mathrm{J}_0\mathrm{H}_1^{-1}\mathrm{J}_0\end{aligned}$$

Lemma A.1.1 yields the existence of a positive invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ such that $\mathrm{W}\mathrm{H}_2\mathrm{W}^* = \mathrm{C}\mathrm{H}_1$.

2. Suppose now that there is an automorphism α on $\mathcal{M}_0$ such that $\alpha(\mathrm{H}_2) = \mathrm{C}\mathrm{H}_1$ for a positive invertible $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$. Since $\mathcal{M}_0$ possesses a cyclic and separating vector there is a unitary W commuting with J_0 such that $\alpha = \mathrm{ad}\,\mathrm{W}$ (see Theorem 2.1.10). Defining $\mathrm{U} := \mathrm{U}_1\mathrm{W}\mathrm{U}_2^*$ we obtain a unitary which obviously commutes with J_0. U also commutes with Δ_0 since

$$\begin{aligned}\mathrm{U}\Delta_0\mathrm{U}^* &= \mathrm{U}_1\mathrm{W}\mathrm{U}_2^*\Delta_0\mathrm{U}_2\mathrm{W}^*\mathrm{U}_1^* \\ &= \mathrm{U}_1\mathrm{W}\mathrm{H}_2\mathrm{J}_0\mathrm{H}_2^{-1}\mathrm{J}_0\mathrm{W}^*\mathrm{U}_1^* \\ &= \mathrm{U}_1\mathrm{C}\mathrm{H}_1\mathrm{J}_0\mathrm{C}^{-1}\mathrm{H}_1^{-1}\mathrm{J}_0\mathrm{U}_1^* \\ &= \Delta_0.\end{aligned}$$

Moreover, we have $\mathrm{U}\mathcal{M}_2\mathrm{U}^* = \mathcal{M}_1$. Since $(\mathcal{M}_0, u_1)$ has modular objects (Δ_1, J_0), the pair $(\mathrm{W}^*\mathcal{M}_0\mathrm{W} = \mathcal{M}_0, \mathrm{W}^* u_1)$ has modular objects $(\mathrm{W}^*\Delta_1\mathrm{W} = \Delta_2, \mathrm{J}_0)$. Since the cyclic and separating vector is (up to an element affiliated with the center) uniquely determined by the modular objects (see Lemma A.2.1) there is a positive invertible $\mathrm{C}_1\eta\mathcal{Z}(\mathcal{M}_0)$ such that $\mathrm{W}^* u_1 = \mathrm{C}_1 u_2$. Setting

$$\mathrm{D} := \mathrm{U}_2\mathrm{C}_1\mathrm{U}_2^*\eta\mathrm{U}_2\mathcal{Z}(\mathcal{M}_0)\mathrm{U}_2^* = \mathcal{Z}(\mathcal{M}_2)$$

we conclude

$$\mathrm{U}^* u_0 = \mathrm{U}_2\mathrm{W}^*\mathrm{U}_1^* u_0 = \mathrm{U}_2\mathrm{W}^* u_1 = \mathrm{U}_2\mathrm{C}_1 u_2 = \mathrm{U}_2\mathrm{C}_1\mathrm{U}_2^*\mathrm{U}_2 u_2 = \mathrm{D} u_0.$$

Thus, $\mathrm{U} \in \mathcal{W}_1$ which proves $\mathcal{M}_1 \sim \mathcal{M}_2$. □

Remark 6.1.2. 1. We can assume without loss of generality that $\mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}) = \mathrm{I}$ for the operator $\mathrm{H}\eta\mathcal{M}_0$ corresponding to an equivalence class of the inverse problem. Indeed, suppose with the above notations that $\mathrm{C} := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}) \neq \mathrm{I}$. As in the beginning of this section we set $\tilde{u} := \mathrm{C}^{-1/2} u$ and get a cyclic and separating vector which has the same modular objects as u. Hence, there is a unitary U_0 commuting with J_0 such that $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{U}_0^*\Delta_0\mathrm{U}_0$ and $\mathrm{U}_0^* u_0 = \tilde{u}$ (see Lemma 5.2.7). Proposition 5.2.3 now leads to

$$\tilde{\mathcal{M}} := \mathrm{U}_0\mathcal{M}_0\mathrm{U}_0^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0).$$

Setting $\mathrm{V} := \mathrm{U}_0\mathrm{U}^*$ we get a unitary such that $\tilde{\mathcal{M}} = \mathrm{V}\mathcal{M}\mathrm{V}^*$, V commutes with Δ_0 and J_0, and

$$\mathrm{V}^* u_0 = \mathrm{U}\mathrm{U}_0^* u_0 = \mathrm{U}\tilde{u} = \mathrm{U}\mathrm{C}^{1/2}u = \underbrace{\mathrm{U}\mathrm{C}^{1/2}\mathrm{U}^*}_{\eta\mathcal{Z}(\mathcal{M})} u_0.$$

We have thus shown that $\tilde{\mathcal{M}} \sim \mathcal{M}$. Moreover, if we denote the positive invertible operator corresponding to $\tilde{\mathcal{M}}$ (and $\tilde{u}$) by $\tilde{\mathrm{H}}$ we obtain $\mathrm{tr}_{\mathcal{M}_0}(\tilde{\mathrm{H}}) = \mathrm{I}$.

2. With the notations of Lemma 6.1.1 we assume the operators H_1 and H_2 to be normalized, i. e. $\mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_1) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_2) = \mathrm{I}$. Then

$$\mathrm{tr}_{\mathcal{M}_0} \circ \alpha = (\mathrm{tr}_{\mathcal{M}_0})_{\mathrm{C}} = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{C}\cdot)$$

holds. In fact, since the central trace $\mathrm{tr}_{\mathcal{M}_0}$ is unique up to a positive invertible operator affiliated with the center of $\mathcal{M}_0$ (cf. the remark following Theorem 2.3.10) we have $\mathrm{tr}_{\mathcal{M}_0} \circ \alpha = (\mathrm{tr}_{\mathcal{M}_0})_{\mathrm{D}}$ for a positive invertible operator $\mathrm{D}\eta\mathcal{Z}(\mathcal{M}_0)$. Furthermore, the normalizing condition implies

$$\mathrm{C} = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{C}\mathrm{H}_1) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{W}\mathrm{H}_2\mathrm{W}^*) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{D}\mathrm{H}_2) = \mathrm{D}.$$

3. If $\mathcal{M}_0$ is a type I or finite type II algebra the operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ of Lemma 6.1.1 equals I, because the central trace is uniquely defined in these cases (see Remark 2.3.11).

   The situation changes completely in the type II_∞ case. For instance, if $\mathcal{M}_0$ is a type II_∞ factor, $\mathrm{C} \in \mathbb{R}_{>0}$ is in the fundamental group

$$G(\mathcal{M}_0) = \{\lambda > 0 |\text{ there is an } \alpha \in \mathrm{aut}(\mathcal{M}_0) \text{ with } \mathrm{tr} \circ \alpha = \lambda\, \mathrm{tr}\}$$

   of $\mathcal{M}_0$ which is in general distinct from the trivial group (see e. g. [KR86, Proposition 13.1.10]).

Remark 6.1.3. Lemma 6.1.1 yields a connection between the inverse problem and the conjugacy problem of group actions. Namely, set $\sigma_t^i := \mathrm{ad}\,\Delta_i^{it}$, $i = 1, 2$, where $\Delta_i = \mathrm{H}_i\mathrm{J}_0\mathrm{H}_i^{-1}\mathrm{J}_0$ is the modular operator corresponding to the solution $\mathcal{M}_i \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ $(i = 1, 2)$. If $\mathcal{M}_1 \sim \mathcal{M}_2$ there is an automorphism $\alpha = \mathrm{ad}\,\mathrm{W} \in \mathrm{aut}(\mathcal{M}_0)$ such that

$$\sigma_t^1 = \mathrm{ad}\,\Delta_1^{it} = \mathrm{ad}\,\mathrm{H}_1^{it} = \mathrm{ad}(\mathrm{C}^{-it}\alpha(\mathrm{H}_2^{it})) = \mathrm{ad}\,\alpha(\mathrm{H}_2^{it}) = \alpha \circ \sigma_t^2 \circ \alpha^{-1}$$

for all $t \in \mathbb{R}$. Hence, σ^1 and σ^2 are conjugate.

On the other hand, if $\mathcal{M}_1, \mathcal{M}_2 \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ and σ^1 and σ^2 are conjugate there is an $\alpha \in \mathrm{aut}(\mathcal{M}_0)$ such that

$$\sigma_t^1(\mathrm{M}) = \mathrm{H}_1^{it}\mathrm{M}\mathrm{H}_1^{-it} = \alpha(\mathrm{H}_2^{it}\alpha^{-1}(\mathrm{M})\mathrm{H}_2^{-it}) = \alpha(\mathrm{H}_2^{it})\mathrm{M}\alpha(\mathrm{H}_2^{-it})$$

for all $\mathrm{M} \in \mathcal{M}_0$ and $t \in \mathbb{R}$. Then $\mathrm{H}_1^{it} = \mathrm{C}^{it}\alpha(\mathrm{H}_2^{it})$ with $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ for all $t \in \mathbb{R}$. According to Lemma 6.1.1 this implies $\mathcal{M}_1 \sim \mathcal{M}_2$.

There is a rich literature on conjugacy problems of group actions. The most progress was made in the research on cocycle conjugacy of group actions on hyperfinite factors. Connes' work on the classification of automorphisms on the hyperfinite II_1 factor up to outer conjugacy implies at the same time a classification up to cocycle conjugacy of actions of $\mathbb{Z}$ [Con75]. In the following his results were extended to actions of countable amenable groups [Ocn85, ST89, KST98] and compact abelian groups [JT84, KT92] on all hyperfinite factors. For the conjugacy problem of actions of $\mathbb{Z}$ there seems to be only the classification of periodic automorphisms by Connes [Con77] and some invariants in the general case (e. g. entropy, see [CS75], cf. also [Jon91, § 2.4]). Kawahigashi did some initial steps for the action of $\mathbb{R}$ [Kaw91a, Kaw89, Kaw91b]. But these results are not applicable to the case of modular automorphism groups since they are only concerned with cocycle or stable conjugacy which is always fulfilled for two modular automorphism groups (cf. Theorem 2.1.15).

We get a characterization of the second simple class from §5.3 as a corollary of Lemma 6.1.1:

Corollary 6.1.4. *Let $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ where U commutes with J_0. Let further $(\Delta := \mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ be the modular objects of $(\mathcal{M}_0, \mathrm{U}^*u_0)$. Suppose that $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$ and $\Delta = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$ with $\mathrm{H}_0, \mathrm{H}\eta\mathcal{M}_0$. Then $\mathcal{M} \in NA^2$ if and only if there exist a unitary W_2 and a positive invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ such that $\mathrm{H}_0 = \mathrm{C}\mathrm{W}_2\mathrm{H}^{-1}\mathrm{W}_2^*$, $\operatorname{ad}\mathrm{W}_2 \in \operatorname{aut}\mathcal{M}_0$, and W_2 commutes with J_0.*

Corollary 6.1.5. *Let $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$ be the decomposition of the modular operator Δ_0. Suppose*

$$NA^1(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) = NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0).$$

Then there exist a unitary $\mathrm{V} \in L(\mathcal{H})$ and a positive invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M}_0)$ such that $\mathrm{H}_0 = \mathrm{C}\mathrm{V}\mathrm{H}_0^{-1}\mathrm{V}^*$, $\operatorname{ad}\mathrm{V} \in \operatorname{aut}\mathcal{M}_0$, and V commutes with J_0.

Proof. Let $\mathcal{M} \in NA^1 = NA^2$ and suppose U be a unitary such that $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$ and

$$\Delta := \mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$$

for a positive invertible operator $\mathrm{H}\eta\mathcal{M}_0$. Then Lemma 6.1.1 and Corollary 6.1.4 imply

$$\mathrm{H}_0 = \mathrm{C}_1\mathrm{W}_1\mathrm{H}\mathrm{W}_1^* \quad \text{and}$$
$$\mathrm{H}_0 = \mathrm{C}_2\mathrm{W}_2\mathrm{H}^{-1}\mathrm{W}_2^*.$$

This shows

$$\mathrm{H}_0 = \mathrm{C}_1\mathrm{W}_1\mathrm{W}_2^*\mathrm{C}_2\mathrm{W}_2\mathrm{W}_1^*\mathrm{W}_1\mathrm{W}_2^*\mathrm{H}_0^{-1}\mathrm{W}_2\mathrm{W}_1^*$$

and with $\mathrm{V} := \mathrm{W}_1\mathrm{W}_2^*$ and $\mathrm{C} := \mathrm{C}_1\mathrm{V}\mathrm{C}_2\mathrm{V}^*$ the assertion follows. ☐

Remark 6.1.6. Later we will show in the case of type I factors that there is a special class of modular operators with so-called *generic spectrum* such that NA^1 and NA^2 are the only classes of solutions of the inverse problem. In contrast, if we consider type II algebras there are always more classes for modular operators Δ_0 corresponding to cyclic and separating vectors which are diagonalizable with respect to the center, provided $\Delta_0 \neq \mathrm{I}$.

According to Lemma 6.1.1 we must look for a complete set of invariants of selfadjoint operators under automorphisms in order to characterize the equivalence classes of "$\sim$". In general, such a set is not known. But if we restrict our considerations to type I algebras or, in the type II case, to vectors diagonalizable with respect to the center (see Definition 4.3.1) we can find such a set in terms of the central spectrum and the central multiplicities introduced in Chapter 4.

6.2 Classification of the Solutions in the Type I Case

In this section we study the classification of the solutions of the inverse problem in the type I case. If the positive invertible operator H is affiliated with a type I_n algebra $\mathcal{M}_0$ ($n \in \mathbb{N} \cup \{\infty\}$) and has finite trace $\mathrm{tr}(\mathrm{H}) < \infty$ we have the following decomposition:

$$\mathrm{H} = \sum_{k \in K} f_k \mathrm{E}_k \tag{6.2.1}$$

where $(\mathrm{E}_k)_{k \in K}$ is a family of pairwise orthogonal projections with sum I and $(f_k)_{k \in K}$ is a family of positive measurable functions affiliated with $\mathcal{Z}(\mathcal{M}_0)$ (see Theorem 4.2.4). Moreover, if $tr_{\mathcal{M}_0}$ is the unique central trace on $\mathcal{M}_0$ (cf. Remark 6.1.2) and $\mathrm{tr} = \omega_\mu \circ \mathrm{tr}_{\mathcal{M}_0}$ then $m_k := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k) \eta \mathcal{Z}(\mathcal{M}_0)$ is almost everywhere integer valued, $\sum_k \omega_\mu(m_k f_k) < \infty$, and the supports of the functions f_k equal the central carriers of the corresponding E_k.

As in §4.2 we treat the factor case separately. Recall that then $f_k \in \mathbb{R}_{>0}$, $m_k \in \mathbb{N}$, and (6.2.1) is the usual spectral decomposition of a trace class operator. Moreover, we have $\sum_k f_k m_k < \infty$.

We start with the following

Definition 6.2.1. Let $\mathcal{M}_0$ be a type I_n von Neumann algebra ($n \in \mathbb{N} \cup \{\infty\}$) and let $\mathrm{H}, \mathrm{H}_1, \mathrm{H}_2 \eta \mathcal{M}_0$ be positive invertible operators which have finite trace.

1. Assume that $\mathrm{H} = \sum_{k \in K} f_k \mathrm{E}_k$ is the decomposition of H. For $k \in K$ we call f_k a *central eigenvalue* of H and $m_k := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k)$ its *central multiplicity.*

2. Assume that $\mathrm{H}_1 = \sum_{k \in K} f_k \mathrm{E}_k$ and $\mathrm{H}_2 = \sum_{l \in L} g_l \mathrm{F}_l$. We call H_1 and H_2 *center-spectrally equivalent* if the index sets K and L have the same cardinality, $f_k = \beta(g_k)$ with an automorphism $\beta \in \mathrm{aut}(\mathcal{Z}(\mathcal{M}_0))$, and $\mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{F}_k)$ for all $k \in K = L$.

Remark 6.2.2. If $\mathcal{M}_0$ is a type I_n factor, i. e. $\mathcal{M}_0 \simeq L(\mathcal{H}_0)$ with an n-dimensional Hilbert space $\mathcal{H}_0$, Definition 6.2.1 coincides with the usual definition of eigenvalues of selfadjoint trace-class operators in $L(\mathcal{H}_0)$ and their multiplicities. The automorphism β is the identity and two positive operators are center-spectrally equivalent if and only if they have the same spectrum and corresponding multiplicities, thus, if and only if they are unitarily equivalent.

The last statement of the preceding remark can also be generalized to the non-factor case.

Proposition 6.2.3. *Let* $\mathrm{H}_1, \mathrm{H}_2 \eta \mathcal{M}_0$ *be two positive invertible operators with finite trace* $\mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_1) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_2) = \mathrm{I}$. *Suppose that they are center-spectrally equivalent. Then there exists an automorphism* $\alpha \in \mathrm{aut}(\mathcal{M}_0)$ *such that*

$$\mathrm{H}_2 = \alpha(\mathrm{H}_1).$$

Proof. By assumption, we can write

$$\begin{aligned} \mathrm{H}_1 &= \sum_{k \in K} f_k \mathrm{E}_k \quad \text{and} \\ \mathrm{H}_2 &= \sum_{k \in K} g_k \mathrm{F}_k \end{aligned}$$

where $(\mathrm{E}_k)_{k \in K}$, $(\mathrm{F}_k)_{k \in K} \in \mathcal{M}_0$ are families of pairwise orthogonal projections in $\mathcal{M}_0$ and $(f_k)_{k \in K}$, $(g_k)_{k \in K}$ are families of measurable functions affiliated with $\mathcal{Z}(\mathcal{M}_0)$. Since H_1 and H_2 have the same central eigenvalues there is an automorphism $\beta \in \mathrm{aut}(\mathcal{Z}(\mathcal{M}_0))$ such that $g_k = \beta(f_k)$ for all $k \in K$.

Setting $\mathrm{G}_k := (\mathrm{id} \otimes \beta)^{-1}(\mathrm{F}_k)$ (note that $\mathcal{M}_0$ is isomorphic to $L(\mathcal{H}_0) \otimes \mathcal{Z}(\mathcal{M}_0)$, see Theorem 4.2.1) we obtain $\mathrm{tr}_{\mathcal{M}_0}(\mathrm{G}_k) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{F}_k) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k)$ since H_1 and H_2 have the same central multiplicities. It follows that there are partial isometries $\mathrm{W}_k \in \mathcal{M}_0$ such that $\mathrm{W}_k^* \mathrm{W}_k = \mathrm{E}_k$ and $\mathrm{W}_k \mathrm{W}_k^* = \mathrm{G}_k$ (cf. (2.3.2)). We define a unitary in $\mathcal{M}_0$ by $\mathrm{U} := \sum_{k \in K} \mathrm{W}_k$ and an automorphism on $\mathcal{M}_0$ by

$$\alpha := (\mathrm{id} \otimes \beta) \circ \mathrm{ad}\, \mathrm{U}.$$

Hence, we have

$$\begin{aligned} \alpha(f_i \mathrm{E}_i) &= (\mathrm{id} \otimes \beta) \sum_{k,l \in K} (\mathrm{W}_k f_i \mathrm{E}_i \mathrm{W}_l^*) \\ &= \beta(f_i)(\mathrm{id} \otimes \beta)(\mathrm{W}_i \mathrm{E}_i \mathrm{W}_i^*) \\ &= g_i (\mathrm{id} \otimes \beta)(\mathrm{G}_i) = g_i \mathrm{F}_i \qquad (i \in K) \end{aligned}$$

which completes the proof. ☐

Lemma 6.1.1 and Proposition 6.2.3 immediately imply the following

Lemma 6.2.4. *Let* (Δ_0, J_0) *be the modular objects of* $(\mathcal{M}_0, u_0)$. *Suppose that* $\mathcal{M}_1, \mathcal{M}_2 \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ *are two solutions of the inverse problem with corresponding positive invertible operators* H_1 *and* H_2. *If* H_1 *and* H_2 *are center-spectrally equivalent then* $\mathcal{M}_1 \sim \mathcal{M}_2$.

Proof. Let α be the automorphism which exists according to Proposition 6.2.3. Then $\mathrm{H}_1 = \alpha(\mathrm{H}_2)$, and the assertion follows by Lemma 6.1.1. ☐

The converse is established by the next

Lemma 6.2.5. *Adopt the notations of Lemma 6.2.4. Suppose that $\mathcal{M}_1 \sim \mathcal{M}_2$. Then* H_1 *and* H_2 *are center-spectrally equivalent.*

Proof. According to Lemma 6.1.1 and Remark 6.1.2 there is an automorphism $\alpha = \operatorname{ad} \mathrm{W}$ of $\mathcal{M}_0$ such that $\mathrm{H}_1 = \mathrm{W}\mathrm{H}_2\mathrm{W}^*$. Since

$$\begin{aligned} \mathrm{H}_1 &= \sum_{k\in K} f_k \mathrm{E}_k \\ &= \mathrm{W}\sum_{j\in J}(g_j\mathrm{F}_j)\mathrm{W}^* = \sum_{j\in J}(\mathrm{W}g_j\mathrm{W}^*)(\mathrm{W}\mathrm{F}_j\mathrm{W}^*), \end{aligned}$$

we obtain $J = K$, $f_k = \mathrm{W}g_k\mathrm{W}^*$, and $\mathrm{E}_k = \mathrm{W}\mathrm{F}_k\mathrm{W}^*$ for all $k \in K$ by the uniqueness of the decomposition (see Theorem 4.2.1). Moreover, it follows

$$\operatorname{tr}_{\mathcal{M}_0}(\mathrm{E}_k) = \operatorname{tr}_{\mathcal{M}_0}(\mathrm{W}\mathrm{F}_k\mathrm{W}^*) = \operatorname{tr}_{\mathcal{M}_0}(\mathrm{F}_k).$$

□

The last two lemmas show that the central eigenvalues and multiplicities are actually the claimed complete set of invariants under automorphisms for positive operators with finite trace. We can therefore use them to characterize the equivalence classes of "$\sim$".

Let now $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$ be the modular operator corresponding to a cyclic and separating vector u_0 for a type I_n von Neumann algebra $\mathcal{M}_0$ ($n \in \mathbb{N} \cup \{\infty\}$). Assume that $\mathrm{H}_0 = \sum_{k\in K} f_k\mathrm{E}_k$. According to (4.2.5) we have $\Delta_0 = \sum_{j\in J} g_j\mathrm{F}_j$ where $(\mathrm{F}_j)_{j\in J}$ is a family of pairwise orthogonal projections in $\mathcal{B}' = \mathcal{Z}(\mathcal{M}_0)'$ with sum I and $(g_j)_{j\in J}$ is a family of positive measurable functions affiliated with $\mathcal{B}$. Setting $m_k := \operatorname{tr}_{\mathcal{M}_0}(\mathrm{E}_k)$ and $n_j = \operatorname{tr}_{\mathcal{B}'}(\mathrm{F}_j)$ we get two families of measurable functions affiliated with $\mathcal{B}$ which have the following properties:

- m_k is integer valued almost everywhere,
- $\sum_{k\in K} m_k = n\mathrm{I}$,
- $\sum_{k\in K} \omega_\mu(m_k f_k) < \infty$,
- for every $j \in J$ there exists at least one pair $(k,l) \in K^2$ such that $g_j = f_k f_l^{-1}$, and
- $n_j = \sum_{\{f_k f_l^{-1} = g_j\}} m_k m_l$ for all $j \in J$.

If there are other sequences $(\tilde{f}_k)_{k\in\tilde{K}}$, $(\tilde{m}_k)_{k\in\tilde{K}}$ fulfilling the same conditions as $(f_k)_{k\in K}$, $(m_k)_{k\in K}$ with respect to $(n_j)_{j\in J}$, $(g_j)_{j\in J}$, Corollary 4.2.13 states that there is a cyclic and separating vector u for $\mathcal{M}_0$ such that the corresponding modular operator is $\Delta = \mathrm{H}\mathrm{J}_0\mathrm{H}\mathrm{J}_0$ where $\mathrm{H} = \sum_{k\in\tilde{K}} \tilde{f}_k\tilde{\mathrm{E}}_k$ and $\tilde{m}_k = \operatorname{tr}_{\mathcal{M}_0}(\tilde{\mathrm{E}}_k)$. In addition, Δ is unitarily equivalent to Δ_0. Lemma A.3.5 now implies that there is a unitary U commuting with J_0 such that $\Delta_0 = \mathrm{U}\Delta\mathrm{U}^*$. This unitary fulfils the prerequisites of Lemma 5.2.7. Hence, there exists a unitary U_0 commuting with J_0 such that $\mathrm{U}_0\Delta\mathrm{U}_0^* = \Delta_0$ and $\mathrm{U}_0^* u_0 = u$. This implies $\mathrm{U}_0\mathcal{M}_0\mathrm{U}_0^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ (cf. Proposition 5.2.3). We have thus proved the following

Lemma 6.2.6. *Let $(f_k)_{k\in K}$, $(m_k)_{k\in K}$ be two families of positive functions affiliated with the center $\mathcal{Z}(\mathcal{M}_0)$ of the type I_n algebra $\mathcal{M}_0$ ($n \in \mathbb{N}\cup\{\infty\}$) such that*

$$m_k(p) \in \begin{cases} \{0,\dots,n\} & \text{if } n \in \mathbb{N} \\ \mathbb{N} & \text{if } n = \infty \end{cases} \tag{6.2.2a}$$

almost everywhere,

$$\sum_{k\in K} m_k = n\mathrm{I}, \tag{6.2.2b}$$

and

$$\sum_{k\in K} m_k f_k = \mathrm{I}. \tag{6.2.2c}$$

Suppose that the relations

$$\{g_j | j \in J\} = \{f_k f_l^{-1} | k,l \in K\} \tag{6.2.3}$$

and

$$n_j = \sum_{\{f_k f_l^{-1} = g_j\}} m_k m_l \tag{6.2.4}$$

are fulfilled.

There exists a solution $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ such that $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$ and H has central eigenvalues $(f_k)_{k\in K}$ and central multiplicities $(m_k)_{k\in K}$.

Remark 6.2.7. We again consider the factor case. Then the families $(f_k)_k$ and $(m_k)_k$ are numbers in $\mathbb{R}_{>0}$ and $\mathbb{N}$, respectively. The families $(g_j)_j$ and $(n_j)_j$ are numbers in $\mathbb{R}_{>0}$ and $\mathbb{N}$ as well. These numbers are the usual eigenvalues of Δ_0 and their multiplicities. Accordingly, (6.2.3) and (6.2.4) are relations between numbers.

We can now summarize the results of this section in the following

Theorem 6.2.8. *Let $\mathcal{M}_0$ be a type I_n ($n \in \mathbb{N}\cup\{\infty\}$) von Neumann algebra with cyclic and separating vector u_0 and corresponding modular objects $(\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0, \mathrm{J}_0)$ where the positive invertible operator $\mathrm{H}_0\eta\mathcal{M}_0$ has finite trace.*

1. *Suppose that $\mathcal{M}_1, M_2 \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ are two solutions of the inverse problem with corresponding positive invertible operators $\mathrm{H}_i\eta\mathcal{M}_0$ ($i = 1,2$). $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent if and only if H_1 and H_2 are center-spectrally equivalent.*

2. *A positive invertible operator $\mathrm{H}\eta\mathcal{M}_0$ gives rise to a solution of the inverse problem if and only if its central eigenvalues $(f_k)_{k\in K}$ and central multiplicities $(m_k)_{k\in K}$ satisfy* (6.2.2), (6.2.3), *and* (6.2.4) *with respect to the central eigenvalues $(g_j)_{j\in J}$ and central multiplicities $(n_j)_{j\in J}$ of Δ_0.*

3. *The equivalence classes of "$\sim$" are classified completely by the central spectrum of the corresponding operators, i. e. by families $(f_k)_k$ and $(m_k)_k$ of positive elements affiliated with the center of $\mathcal{M}_0$ which satisfy* (6.2.2), (6.2.3), *and* (6.2.4).

6.3 Classification of the Solutions in the Type II Case

In this section we study the classification of the solutions of the inverse problem in the type II case. Since we will use the results of §4.3 we must restrict ourselves to the case of cyclic and separating vectors diagonalizable with respect to the center (see Definition 4.3.1).

Let u_0 be a cyclic and separating vector for the type II_n algebra $\mathcal{M}_0$ ($n \in \{1, \infty\}$). Assume that u_0 is diagonalizable with respect to the center. Then we have the following decomposition for the positive operator $\mathrm{H}\eta\mathcal{M}_0$ which generates the modular operator corresponding to u_0:

$$\mathrm{H} = \sum_{k \in K} f_k \mathrm{E}_k, \tag{6.3.1}$$

where $(\mathrm{E}_k)_{k\in K}$ is a family of pairwise orthogonal projections with sum I and $(f_k)_{k\in K}$ is a family of positive measurable functions affiliated with $\mathcal{Z}(\mathcal{M}_0)$ (see Definition 4.3.1). Moreover, if $tr_{\mathcal{M}_0}$ is a central trace on $\mathcal{M}_0$ and $\mathrm{tr} = \omega_\mu \circ \mathrm{tr}_{\mathcal{M}_0}$ then $\sum_k \omega_\mu(m_k f_k) < \infty$ with $m_k := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k)\eta\mathcal{Z}(\mathcal{M}_0)$ and the supports of the functions f_k equal the central carriers of the corresponding E_k.

Recall that if $\mathcal{M}_0$ is a factor we have $f_k \in \mathbb{R}_{>0}$, $m_k \in \mathbb{N}$, and (6.3.1) means that H possesses pure point spectrum.

Suppose that $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ is a solution of the inverse problem such that $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$ for an operator $\mathrm{H}\eta\mathcal{M}_0$ which is diagonalizable with respect to the center. Then we call $\mathcal{M}$ a *solution diagonalizable with respect to the center*. By Lemma 6.1.1 this notion is independent of the unitary U such that $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$.

The following definition is the analogue of Definition 6.2.1:

Definition 6.3.1. Let $\mathcal{M}_0$ be a type II_n von Neumann algebra ($n \in \{1, \infty\}$) and $\mathrm{H}, \mathrm{H}_1, \mathrm{H}_2$ be positive invertible operators diagonalizable with respect to the center of $\mathcal{M}_0$.

1. Assume that $\mathrm{H} = \sum_{k\in K} f_k \mathrm{E}_k$ is the decomposition of H. For $k \in K$ we call f_k a *central eigenvalue* of H and $m_k := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k)$ its *central multiplicity*.

2. Assume that $\mathrm{H}_1 = \sum_{k\in K} f_k \mathrm{E}_k$, $\mathrm{H}_2 = \sum_{l\in L} g_l \mathrm{F}_l$, and define $m_k := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k)$, $n_l := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{F}_l)$. We call H_1 and H_2 *center-spectrally equivalent* if the index sets K and L have the same cardinality and there are an automorphism $\beta \in \mathrm{aut}(\mathcal{M}_0)$ and a positive measurable function h affiliated with $\mathcal{Z}(\mathcal{M}_0)$ such that $f_k = \beta(g_k)$, $m_k = hn_k$ and $\mathrm{tr}_{\mathcal{M}_0} \circ \beta = (\mathrm{tr}_{\mathcal{M}_0})_h$ for all $k \in K = L$.

Remark 6.3.2.

1. If $\mathcal{M}_0$ is a type II_1 algebra we can omit the function h since, as in the type I case, the central trace is unique for type II_1 algebras (cf. Remark 6.1.2).

2. Note that we require in Definition 6.3.1 the existence of an automorphism β on $\mathcal{M}_0$ whereas in Definition 6.2.1 we only required the existence of

an automorphism on $\mathcal{Z}(\mathcal{M}_0)$. If we restrict ourselves to the case $\mathcal{M}_0 = \mathcal{F} \otimes \mathcal{A}$ for a type II factor $\mathcal{F}$ and an abelian algebra $\mathcal{A}$, the similarity to Definition 6.2.1 is more stringent, because every automorphism β on $\mathcal{Z}(\mathcal{M}_0) = \mathcal{A}$ is then the restriction of the automorphism $\mathrm{id} \otimes \beta \in \mathrm{aut}(\mathcal{M}_0)$ to $\mathcal{A}$.

3. In the factor case the definition of the central eigenvalues coincides with the usual definition of eigenvalues. The central multiplicities, however, are different from the usual multiplicities also in the factor case in contrast to the type I case (cf. Remark 6.2.2). The usual multiplicities are always infinite for type II factors (cf. Remark 4.3.6).

Proposition 6.3.3. *Let* $\mathrm{H}_1, \mathrm{H}_2 \eta \mathcal{M}_0$ *be two positive operators diagonalizable with respect to the center with finite trace* $\mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_1) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{H}_2) = \mathrm{I}$. *Suppose that they are center-spectrally equivalent. Then there exists an automorphism* $\alpha \in \mathrm{aut}(\mathcal{M}_0)$ *such that* $\mathrm{H}_2 = \alpha(\mathrm{H}_1)$.

Proof. By assumption, we can write

$$\begin{aligned} \mathrm{H}_1 &= \sum_{k \in K} f_k \mathrm{E}_k \quad \text{and} \\ \mathrm{H}_2 &= \sum_{k \in K} g_k \mathrm{F}_k \end{aligned}$$

where $(\mathrm{E}_k)_{k\in K}$, $(\mathrm{F}_k)_{k\in K}$ are families of pairwise orthogonal projections in $\mathcal{M}_0$ and $(f_k)_{k\in K}$, $(g_k)_{k\in K}$ are families of measurable functions affiliated with $\mathcal{Z}(\mathcal{M}_0)$. Since H_1 and H_2 have the same central eigenvalues there is an automorphism $\beta \in \mathrm{aut}(\mathcal{M}_0)$ such that $g_k = \beta(f_k)$ for all $k \in K$.

Setting $\mathrm{G}_k := \beta^{-1}(\mathrm{F}_k)$ we obtain

$$\mathrm{tr}_{\mathcal{M}_0}(\mathrm{G}_k) = (\mathrm{tr}_{\mathcal{M}_0})_{h^{-1}}(\mathrm{F}_k) = h^{-1} h \,\mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k) = \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k)$$

since H_1 and H_2 have the same central multiplicities, i. e. $\mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k) = h\,\mathrm{tr}_{\mathcal{M}_0}(\mathrm{F}_k)$. It follows that there are partial isometries $\mathrm{W}_k \in \mathcal{M}_0$ such that $\mathrm{W}_k^*\mathrm{W}_k = \mathrm{E}_k$ and $\mathrm{W}_k\mathrm{W}_k^* = \mathrm{G}_k$ (cf. (2.3.2)). We define a unitary in $\mathcal{M}_0$ by $\mathrm{U} := \sum_{k\in K} \mathrm{W}_k$ and an automorphism on $\mathcal{M}_0$ by

$$\alpha := \beta \circ \mathrm{ad}\,\mathrm{U}.$$

Hence, we have

$$\begin{aligned} \alpha(f_i \mathrm{E}_i) &= \beta \sum_{k,l \in K} (\mathrm{W}_k f_i \mathrm{E}_i \mathrm{W}_l^*) \\ &= \beta(f_i)\beta(\mathrm{W}_i \mathrm{E}_i \mathrm{W}_i^*) \\ &= g_i \beta(\mathrm{G}_i) = g_i \mathrm{F}_i \qquad (i \in K) \end{aligned}$$

which completes the proof. $\square$

The following two lemmas now follow in exactly the same way as Lemma 6.2.4 and Lemma 6.2.5:

Lemma 6.3.4. *Let* (Δ_0, J_0) *be the modular objects of* $(\mathcal{M}_0, u_0)$*. Suppose that* $\mathcal{M}_1, M_2 \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ *are two solutions of the inverse problem with corresponding positive invertible operators* H_1 *and* H_2*. If* H_1 *and* H_2 *are center-spectrally equivalent then* $\mathcal{M}_1 \sim \mathcal{M}_2$*.*

Lemma 6.3.5. *Adopt the notations of Lemma 6.3.4. Suppose that* $\mathcal{M}_1 \sim \mathcal{M}_2$*. Then* H_1 *and* H_2 *are center-spectrally equivalent.*

The last two lemmas show that the central eigenvalues and multiplicities are the claimed complete set of invariants under automorphisms for positive operators diagonalizable with respect to the center also in the type II case. We can therefore use them to characterize the equivalence classes of "$\sim$".

Let now $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$ be the modular operator corresponding to a cyclic and separating vector u_0 for a type II_n von Neumann algebra $\mathcal{M}_0$ ($n \in \{1, \infty\}$). Assume that $\mathrm{H}_0 = \sum_{k\in K} f_k \mathrm{E}_k$. According to (4.3.1) we have $\Delta_0 = \sum_{j\in J} g_j \mathrm{F}_j$ where $(\mathrm{F}_j)_{j\in J}$ is a family of pairwise orthogonal projections in $\mathcal{B}' = \mathcal{Z}(\mathcal{M}_0)'$ with sum I and $(g_j)_{j\in J}$ is a family of positive measurable functions affiliated with $\mathcal{B}$. Setting $m_k := \mathrm{tr}_{\mathcal{M}_0}(\mathrm{E}_k)$ and $n_j := \mathrm{tr}_{\mathcal{B}'}(\mathrm{F}_j)$ we get two families of measurable functions affiliated with $\mathcal{B}$ which have the following properties:

- $\sum_{k\in K} m_k = n\mathrm{I}$,
- $\sum_{k\in K} \omega_\mu(m_k f_k) < \infty$, and
- for every $j \in J$ there exists at least one pair $(k, l) \in K^2$ such that $g_j = f_k f_l^{-1}$.

The next lemma is proved in exactly the same way as the corresponding Lemma 6.2.6.

Lemma 6.3.6. *Let* $(f_k)_{k\in K}$*,* $(m_k)_{k\in K}$ *be two families of positive functions affiliated with the center* $\mathcal{Z}(\mathcal{M}_0)$ *of the type* II_n *algebra* $\mathcal{M}_0$ *(*$n \in \{1, \infty\}$*) such that*

$$m_k(p) \in \begin{cases} (0, 1] & \text{if } n = 1 \\ \mathbb{R}_{>0} & \text{if } n = \infty \end{cases} \tag{6.3.2a}$$

almost everywhere,

$$\sum_{k\in K} m_k = n\mathrm{I} \tag{6.3.2b}$$

and

$$\sum_{k\in K} m_k f_k = \mathrm{I}. \tag{6.3.2c}$$

Suppose that the relations

$$\{g_j | j \in J\} = \{f_k f_l^{-1} | k, l \in K\} \tag{6.3.3}$$

are fulfilled.

There exists a solution $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ *such that* $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$ *and* H *has central eigenvalues* $(f_k)_{k\in K}$ *and central multiplicities* $(m_k)_{k\in K}$*.*

Remark 6.3.7. Concerning the factor case, Remark 6.2.7 applies here as well.

We can now summarize the results of this section in the following

Theorem 6.3.8. *Let $\mathcal{M}_0$ be a type II_n ($n \in \{1,\infty\}$) von Neumann algebra with cyclic and separating vector u_0 and corresponding modular objects ($\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0, \mathrm{J}_0$) where the positive invertible operator $\mathrm{H}_0 \eta \mathcal{M}_0$ is diagonalizable with respect to the center of $\mathcal{M}_0$.*

1. *Suppose that $\mathcal{M}_1, M_2 \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ are two solutions of the inverse problem which are diagonalizable with respect to the center. They are equivalent if and only if the corresponding positive invertible operators $\mathrm{H}_1 \eta \mathcal{M}_0$ and $\mathrm{H}_2 \eta \mathcal{M}_0$ are center-spectrally equivalent.*

2. *A positive invertible operator $\mathrm{H} \eta \mathcal{M}_0$ gives rise to a solution of the inverse problem if and only if its central eigenvalues $(f_k)_{k\in K}$ and central multiplicities $(m_k)_{k\in K}$ satisfy* (6.3.2) *and* (6.3.3) *with respect to the central eigenvalues $(g_j)_{j\in J}$ and central multiplicities $(n_j)_{j\in J}$ of Δ_0.*

3. *The equivalence classes of "$\sim$" are classified completely by the central spectrum of the corresponding operators, i. e. by families $(f_k)_k$ and $(m_k)_k$ of positive elements affiliated with the center of $\mathcal{M}_0$ which satisfy* (6.3.2) *and* (6.3.3).

Example 6.3.9. The following examples illustrate Theorem 6.3.8. To avoid technical difficulties we restrict our attention to the factor case. The central eigenvalues are hence the usual eigenvalues, and the central multiplicities are numbers in $\mathbb{R}_{>0}$.

1. Let $(\mu_k)_{k\in K}$ and $(m_k)_{k\in K}$ be the central eigenvalues and multiplicities of a positive operator H_0 affiliated with a type II_1 factor $\mathcal{M}_0$. Suppose that $(\mu_k)_k$ and $(m_k)_k$ as well as $(c\mu_k^{-1})_k$ and $(m_k)_k$ in place of $(\mu_k)_k$ and $(m_k)_k$ fulfil the conditions (6.3.2) where $c > 0$ is an appropriately chosen constant. This implies that $\Delta_0^{-1} = \mathrm{H}_0^{-1}\mathrm{J}_0\mathrm{H}_0\mathrm{J}_0$ is also a modular operator for $\mathcal{M}_0$, the equivalence class $NA^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ exists, and it is characterized by the families $(c\mu_k^{-1})_k$ and $(m_k)_k$. If there is a permutation σ of K such that $(c\mu_{\sigma(k)}^{-1})_k = (\mu_k)_k$ and $(m_{\sigma(k)})_k = (m_k)_k$ this class is the first simple class, i.e. $NA^2 = NA^1$ (cf. Corollary 6.1.5).

2. In contrast to the type I case, the index set K can also be countably infinite for finite type II factors (and it is always infinite for infinite type II factors): Set $\mu_k = 15/(k\pi)^2$ and $m_k = 6/(k\pi)^2$ for $k \in \mathbb{N}$. Then $\sum_{k\in\mathbb{N}} m_k = 1$ and $\sum_{k\in\mathbb{N}} \mu_k m_k = 1$. Hence, the countably infinite families $(\mu_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ fulfil (6.3.2).

3. Let
$$(10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3)$$
be the eigenvalues of a modular operator for a type II_1 factor. Then
$$((c_1 \cdot 1, 1/4), (c_1 \cdot 10^{-1}, 1/4), (c_1 \cdot 10^{-2}, 1/4), (c_1 \cdot 10^{-3}, 1/4)),$$

$$((c_2 \cdot 10^3, 1/4), (c_2 \cdot 10^2, 1/4), (c_2 \cdot 10^1, 1/4), (c_2 \cdot 1, 1/4)),$$
$$((c_3 \cdot 1, 1/3), (c_3 \cdot 10^{-1}, 1/3), (c_3 \cdot 10^{-3}, 1/3)),$$

and

$$((c_4 \cdot 10^3, 1/3), (c_4 \cdot 10^1, 1/3), (c_4 \cdot 1, 1/3))$$

characterize four different classes of solutions of the inverse problem where c_i $(i = 1, 2, 3, 4)$ are appropriately chosen constants. This implies that there are more classes of solutions than the two simple ones NA^1 and NA^2.

4. Let
$$(\ldots, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3, \ldots)$$
be the eigenvalues of a modular operator for a type II_∞ factor. Then
$$((c_1 \cdot 1, 1), (c_1 \cdot 10^{-1}, 1), (c_1 \cdot 10^{-2}, 1), (c_1 \cdot 10^{-3}, 1), \ldots)$$
and
$$((c_2 \cdot 1, 1), (c_2 \cdot 10^{-1}, 1), (c_2 \cdot 10^{-3}, 1), (c_2 \cdot 10^{-5}, 1), \ldots)$$
characterize two different classes of solutions of the inverse problem, i. e. they both satisfy (6.3.3) and (6.3.2) where c_i $(i = 1, 2)$ are appropriately chosen constants. This implies that there are also more classes of solutions than the simple class NA^1 in the type II_∞ case.

5. Assume that the families $(\mu_k)_{k \in K}$ and $(m_k)_{k \in K}$ characterize a class of solutions of the inverse problem for a type II factor. Suppose that $m_l \neq m_k$ for at least one pair $k, l \in K$. Let σ be a finite permutation of K interchanging k and l. Then $(c\mu_k)_k$ and $(m_{\sigma(k)})_k$ characterize another class of solutions of the inverse problem ($c > 0$ is a normalizing constant). If $|K| = 2$ and $\mu_1 = \mu_2^{-1}$, this is again the second simple class NA^2, otherwise it is a different one.

6. Assume again that the families $(\mu_k)_{k \in K}$ and $(m_k)_{k \in K}$ characterize a solution of the inverse problem for a type II factor. Let $k, l \in K$ be two indices and fix $\epsilon > 0$. Adding ϵ to m_k and subtracting it from m_l, we get another class of solutions. This is again the the second simple class NA^2 if $|K| = 2$, $\mu_1 = \mu_2^{-1}$, and $m_1 = m_2 - \epsilon$.

Remark 6.3.10. Example 6.3.9.5 and Example 6.3.9.6 show the following fact: Let $\mathcal{M}_0$ be a type II factor. If $\mathrm{H}_0 \eta \mathcal{M}_0$ has more than one eigenvalue, which implies that the corresponding modular operator Δ_0 is not the identity (the case $\Delta_0 = \mathrm{I}$ was treated in Example 5.2.6), we can always construct a class of solutions which is different from the two simple classes discussed in §6.1, i. e. $NA \neq NA^1 \cup NA^2$.

Unfortunately, the classification result presented here applies only to cyclic and separating vectors which are diagonalizable with respect to the center. In general, there are also vectors with more complicated spectrum (cf. §4.3.2 and Example 4.3.3 therein).

In order to treat the general case, there are three different problems to solve which we present exemplarily for the type II_1 factor case: Let H_1 and H_2 be two positive invertible operators affiliated with a type II_1 factor $\mathcal{M}_0$.

1. **Under which conditions are H_1 and H_2 unitarily equivalent?**

   This question is solved with the help of the usual spectrum and multiplicity function of selfadjoint operators (see e. g. [BS87]).

2. Assume that H_1 and H_2 are unitarily equivalent and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{M}_0$ be the abelian von Neumann algebras generated by H_1 and H_2. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic as von Neumann algebras.

   Under which conditions are $\mathcal{A}_1$ and $\mathcal{A}_2$ conjugate to each other (with respect to $\mathcal{M}_0$)?

   If both are Cartan subalgebras of $\mathcal{M}_0$ and if $\mathcal{M}_0$ is the hyperfinite factor of type II_1 they are always conjugate by the theorem of Connes, Feldman, and Weiss (cf. §4.3.2). For an arbitrary maximal abelian subalgebra $\mathcal{A}$ of a type II_1 factor there are some conjugacy invariants, for instance the type of the commutant of the algebra $\mathcal{A} \vee \mathrm{J}\mathcal{A}\mathrm{J}$ where J is the modular conjugation with respect to a trace vector. It is known, however, that this invariant does not suffice to determine whether two maximal abelian subalgebras are conjugate (see [Puk60] and [Pop85]). It is also an open question whether this invariant is sufficient to classify at least the singular maximal abelian subalgebras. For another invariant see also [Pop83].

3. Let $\mathcal{A} \subset \mathcal{M}_0$ be an abelian subalgebra and $\alpha \in \mathrm{aut}(\mathcal{A})$ an automorphism on $\mathcal{A}$.

   Under which conditions is α the restriction of an automorphism on $\mathcal{M}_0$ to $\mathcal{A}$?

   A necessary condition for a type II_1 factor is the invariance of the trace under the automorphism α. Up to now, it is not clear if this is also sufficient.

In the diagonalizable case all these question were answered: Two operators are unitarily equivalent if and only if they have equal spectrum (the multiplicities are always infinite, see Remark 4.3.6). Furthermore, the corresponding abelian algebras are generated by countably many (minimal) projections. They are hence conjugate if and only if there is a bijective mapping between the two generating sets of projections which assigns to each projection an equivalent one (this is a part of Proposition 6.3.3). Finally, an automorphism on a countable generated abelian subalgebra is implemented by an automorphism on the superalgebra if and only if it maps the generating projections onto equivalent ones, i. e. with the same trace (this is also Proposition 6.3.3). The above problems are thus solved for the diagonalizable case if we use the central spectrum and the central multiplicities defined in Definition 6.3.1.

6.4 Remarks on the Type III_λ Case ($0 < \lambda < 1$)

In this section we make some remarks on the classification of the solutions of the inverse problem in the type III_λ case ($0 < \lambda < 1$). Up to now, it is not possible to obtain a theorem similar to Theorem 6.2.8 and Theorem 6.3.8. The lemmas of this section only contain some preliminary results. As in §6.3 we restrict our attention to the case of cyclic and separating vectors diagonalizable with respect to the center (see §4.4).

Let now (Δ_0, J_0) be the modular objects for a type III_λ factor $\mathcal{M}_0$ ($0 < \lambda < 1$) corresponding to the cyclic and separating vector u_0. Let further $\mathcal{M}_0 = \mathcal{R}(\mathcal{N}, \theta)$ be the discrete decomposition of type III_λ such that u_0 is the dual vector of a cyclic and separating vector for the type II_∞ factor $\mathcal{N}$ (cf. Proposition 3.4.1). Then $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\Delta_{\mu_0}$ where $\mathrm{H}_0 \in \mathcal{N}$ and Δ_{μ_0} is the modular operator corresponding to a cyclic and separating generalized vector μ_0 which generates the dual τ of a trace on $\mathcal{N}$ (see Theorem 3.4.6). As in §4.4 we assume that $\mathrm{J}_0 = \mathrm{J}_{\mu_0}$ where the latter is the modular conjugation corresponding to μ_0 (cf. also §3.4).

If a factor $\mathcal{M}$ is a solution of the inverse problem there is a unitary U such that $\mathrm{U}\mathcal{M}_0\mathrm{U}^* = \mathcal{M}$ and $(\Delta := \mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ are the modular objects for $(\mathcal{M}_0, u := \mathrm{U}^*u_0)$ (see Proposition 5.2.3). According to Proposition 3.4.1 there is a discrete decomposition of type III_λ, $\mathcal{M}_0 = \mathcal{R}(\tilde{\mathcal{N}}, \tilde{\theta})$, such that u is the dual vector of a cyclic and separating vector for the type II_∞ factor $\tilde{\mathcal{N}}$ and $\Delta = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0\Delta_{\tilde{\mu}_0}$ where $\tilde{\mu}_0$ is also a cyclic and separating generalized vector which generates the dual of a trace on $\tilde{\mathcal{N}}$. Assume again $\mathrm{J}_0 = \mathrm{J}_{\tilde{\mu}_0}$. In general, $\mu_0 \neq \tilde{\mu}_0$ and thus $\Delta_{\mu_0} \neq \Delta_{\tilde{\mu}_0}$. But the next lemma states that we can assume without loss of generality that $\tilde{\mu}_0$ coincides with μ_0, and hence $\tilde{\mathcal{N}} = \mathcal{N}$ and $\Delta_{\mu_0} = \Delta_{\tilde{\mu}_0}$:

Lemma 6.4.1. *Adopt the above notations. Let* $(\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\Delta_{\mu_0}, \mathrm{J}_0)$ *be the modular objects of* $(\mathcal{M}_0, u_0)$*. Let further* $\mathcal{M} \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ *be a solution of the inverse problem. Then there is a unitary* U *commuting with* J_0 *such that* $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$ *and the vector* U^*u_0 *is cyclic and separating for* $\mathcal{M}_0$ *with modular objects* $(\mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0\Delta_{\mu_0} = \mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$ *where* $\mathrm{H} \in \mathcal{N}$.

Proof. Since $\mathcal{M} \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ there is, according to Proposition 5.2.3, a unitary $\tilde{\mathrm{U}}$ commuting with J_0 such that $\mathcal{M} = \tilde{\mathrm{U}}\mathcal{M}_0\tilde{\mathrm{U}}^*$ and $(\tilde{\Delta} := \tilde{\mathrm{U}}^*\Delta_0\tilde{\mathrm{U}}, \mathrm{J}_0)$ are the modular objects of $(\mathcal{M}_0, u := \tilde{\mathrm{U}}^*u_0)$. According to §3.4.1 there are a generalized trace $\tilde{\tau}$, a type II_∞ factor $\tilde{\mathcal{N}} = \mathcal{M}_0^{\tilde{\tau}}$, and a positive invertible operator $\tilde{\mathrm{H}} \in \tilde{\mathcal{N}}$ such that $\tilde{\Delta} = \tilde{\mathrm{H}}\mathrm{J}_0\tilde{\mathrm{H}}^{-1}\mathrm{J}_0\Delta_{\tilde{\mu}_0}$ where $\Delta_{\tilde{\mu}_0}$ is the modular operator corresponding to a cyclic and separating generalized vector $\tilde{\mu}_0$ which generates $\tilde{\tau}$.

By the (essential) uniqueness of the generalized trace, there exist a unitary $\mathrm{V} \in \mathcal{M}_0$ and a $\mu > 0$ such that $\tilde{\tau} = \mu(\tau \circ \mathrm{ad}\,\mathrm{V})$ (cf. e.g. [Str81, Proposition 29.5]). Without loss of generality we can assume $\mu = 1$. Defining $\mathrm{W} := \mathrm{V}\mathrm{J}_0\mathrm{V}\mathrm{J}_0$ we obtain a unitary such that $\Delta_{\tilde{\mu}_0} = \mathrm{W}\Delta_{\mu_0}\mathrm{W}^*$, which is shown by a simple calculation, and $\tilde{\mathcal{N}} = \mathrm{W}\mathcal{N}\mathrm{W}^*$. Setting $\mathrm{U} := \tilde{\mathrm{U}}\mathrm{W}$ we conclude that $\mathrm{U}\mathcal{M}_0\mathrm{U}^* = \mathcal{M}$ and U commutes with J_0. Since $(\mathcal{M}, u_0)$ has modular objects

(Δ_0, J_0), the pair $(\mathcal{M}_0 = \mathrm{U}^*\mathcal{M}\mathrm{U}, \mathrm{U}^* u_0)$ has modular objects $(\mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$. Now

$$\begin{aligned} \mathrm{U}^*\Delta_0\mathrm{U} &= \mathrm{W}^*\tilde{\mathrm{U}}^*\Delta_0\tilde{\mathrm{U}}\mathrm{W} \\ &= \mathrm{W}^*\tilde{\mathrm{H}}\mathrm{J}_0\tilde{\mathrm{H}}^{-1}\mathrm{J}_0\Delta_{\tilde{\mu}_0}\mathrm{W} \\ &= \mathrm{W}^*\tilde{\mathrm{H}}\mathrm{W}\mathrm{J}_0\mathrm{W}^*\tilde{\mathrm{H}}^{-1}\mathrm{W}\mathrm{J}_0\mathrm{W}^*\Delta_{\tilde{\mu}_0}\mathrm{W} \\ &= \underbrace{\mathrm{W}^*\tilde{\mathrm{H}}\mathrm{W}}_{=:\mathrm{H}\in\mathcal{N}}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0\Delta_{\mu_0} \end{aligned}$$

which proves the lemma. □

In the remainder of this section we restrict the discussion to the case of cyclic and separating vectors which are diagonalizable with respect to the center (see Definition 4.3.1). We can therefore apply the results of §4.4. The next lemma states that we can reduce partially the inverse problem for type III_λ factors to the inverse problem for type II_∞ factors.

Lemma 6.4.2. *Let $\mathcal{M}_0$ be a type III_λ factor ($0 < \lambda < 1$) and $u_0 \in \mathcal{H}$ be a cyclic and separating vector which is diagonalizable with respect to the center. Let $\mathcal{R}(\mathcal{N}, \theta)$ be the discrete decomposition of $\mathcal{M}_0$ corresponding to u_0 and $(\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\Delta_{\mu_0}, \mathrm{J}_0)$ be the modular objects of u_0 where $\mathrm{H}_0 \in \mathcal{N}$. Let further $v_0 \in \mathcal{H}_0$ be the corresponding cyclic and separating vector for $\mathcal{N}$ and $(\Delta = \mathrm{H}_0\mathrm{J}\mathrm{H}_0^{-1}\mathrm{J}, \mathrm{J})$ be its modular objects.*

(i) Every solution of the inverse problem for the modular objects (Δ, J) of $(\mathcal{N}, v_0)$ which is diagonalizable with respect to the center gives rise to a solution of the inverse problem for the modular objects (Δ_0, J_0) of $(\mathcal{M}_0, u_0)$ which is also diagonalizable with respect to the center.

(ii) Furthermore, two equivalent solutions of the inverse problem for the type II_∞ factor $\mathcal{N}$ result in equivalent solutions of the inverse problem for the type III_λ factor $\mathcal{M}_0$.

Proof. (i) Let $\tilde{\mathcal{N}} \in NA(\Delta, \mathrm{J}, v_0; \mathcal{N})$ be a solution of the inverse problem for the modular objects (Δ, J) of $(\mathcal{N}, v_0)$ which is diagonalizable with respect to the center. By Proposition 5.2.3, there exists a unitary $\mathrm{U} \in \mathcal{U}(\mathcal{H}_0)$ commuting with J such that $\tilde{\mathcal{N}} = \mathrm{U}\mathcal{N}\mathrm{U}$ and U^*v_0 is cyclic and separating for $\mathcal{N}$ with modular objects $(\tilde{\Delta} = \mathrm{U}^*\Delta\mathrm{U} = \tilde{\mathrm{H}}\mathrm{J}\tilde{\mathrm{H}}^{-1}\mathrm{J}, \mathrm{J})$. Assume that $\tilde{\mathrm{H}} = \sum_{k\in K} f_k\mathrm{E}_k$ is the decomposition of $\tilde{\mathrm{H}}$.

As in the proof of Theorem 4.4.7 we can show that there is a cyclic and separating vector u for $\mathcal{M}_0$ such that the corresponding modular operator Δ_1 equals $\tilde{\mathrm{H}}\mathrm{J}_0\tilde{\mathrm{H}}^{-1}\mathrm{J}_0\Delta_{\mu_0}$. Since the sequences $(f_k)_{k\in K}$ and $(m_k := \mathrm{tr}_{\mathcal{N}}(\mathrm{E}_k))_{k\in K}$ fulfil the prerequisites of Corollary 4.4.8 with respect to Δ_0, the operator Δ_1 is unitarily equivalent to Δ_0. Thus, by Lemma A.3.5, there exists a unitary U_1 commuting with J_0 such that $\Delta_0 = \mathrm{U}_1\Delta_1\mathrm{U}_1^*$. The prerequisites of Lemma 5.2.7 are thus fulfilled and there is a unitary U_0 commuting with J_0 such that $\mathrm{U}_0\Delta_1\mathrm{U}_0^* = \Delta_0$ and $\mathrm{U}_0^*u_0 = u$. This implies $\mathrm{U}_0\mathcal{M}_0\mathrm{U}_0^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ (cf. Proposition 5.2.3).

(ii) Let now $\mathcal{N}_1$ and $\mathcal{N}_2$ be two equivalent solutions of the type II_∞ problem, which are diagonalizable with respect to the center, and let $\mathcal{M}_1$, $\mathcal{M}_2$ denote the corresponding solutions for the type III_λ problem. Hence, there are two unitaries $\tilde{\mathrm{U}}_1$, $\tilde{\mathrm{U}}_2$ commuting with J such that $\tilde{\mathrm{U}}_i^* v_0$, $i = 1, 2$, is cyclic and separating for $\mathcal{N}$ and has modular objects $(\mathrm{H}_i \mathrm{J} \mathrm{H}_i^{-1} \mathrm{J}, \mathrm{J})$ (cf. Proposition 5.2.3). Furthermore, there are two unitaries U_1, U_2 commuting with J_0 such that $\mathrm{U}_i^* u_0$ is cyclic and separating for $\mathcal{M}_0$ and has modular objects $(\mathrm{H}_i \mathrm{J}_0 \mathrm{H}_i^{-1} \mathrm{J}_0 \Delta_{\mu_0}, \mathrm{J}_0)$ $(i = 1, 2)$ (cf. Lemma 6.4.1).

Since $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent the operators $\mathrm{H}_1, \mathrm{H}_2 \in \mathcal{N}$ are conjugate in $\mathcal{N}$. More precisely, the proof of Proposition 6.3.3 implies that there is a unitary $\mathrm{V} \in \mathcal{N} \subset \mathcal{M}_0$ such that $\mathrm{H}_1 = \mathrm{V}\mathrm{H}_2\mathrm{V}^*$. Setting $\mathrm{W} := \mathrm{V}\mathrm{J}_0\mathrm{V}\mathrm{J}_0$ and $\mathrm{U} := \mathrm{U}_1 \mathrm{W} \mathrm{U}_2^*$ we infer that the unitary U commutes with J_0 and $\mathrm{U}\mathcal{M}_2\mathrm{U}^* = \mathcal{M}_1$. Furthermore, U commutes with Δ_0 since

$$\begin{aligned}
\mathrm{U}\Delta_0\mathrm{U}^* &= \mathrm{U}_1\mathrm{W}\mathrm{U}_2^*\Delta_0\mathrm{U}_2\mathrm{W}^*\mathrm{U}_1^* \\
&= \mathrm{U}_1\mathrm{V}\mathrm{J}_0\mathrm{V}\mathrm{J}_0\mathrm{H}_2\mathrm{J}_0\mathrm{H}_2^{-1}\mathrm{J}_0\Delta_{\mu_0}\mathrm{J}_0\mathrm{V}^*\mathrm{J}_0\mathrm{V}^*\mathrm{U}_1^* \\
&= \mathrm{U}_1\mathrm{V}\mathrm{H}_2\mathrm{J}_0\mathrm{V}\mathrm{H}_2^{-1}\mathrm{V}^*\mathrm{J}_0\mathrm{V}^*\Delta_{\mu_0}\mathrm{U}_1^* \\
&= \mathrm{U}_1\mathrm{V}\mathrm{H}_2\mathrm{V}^*\mathrm{J}_0\mathrm{V}\mathrm{H}_2^{-1}\mathrm{V}^*\mathrm{J}_0\Delta_{\mu_0}\mathrm{U}_1^* \\
&= \mathrm{U}_1\mathrm{H}_1\mathrm{J}_0\mathrm{H}_1^{-1}\mathrm{J}_0\Delta_{\mu_0}\mathrm{U}_1^* \\
&= \mathrm{U}_1\Delta_1\mathrm{U}_1^* = \Delta_0.
\end{aligned}$$

Now, $(\mathrm{W}^*\mathcal{M}_0\mathrm{W} = \mathcal{M}_0, \mathrm{W}^*u_1)$ has modular objects $(\mathrm{W}^*\Delta_1\mathrm{W} = \Delta_2, \mathrm{J}_0)$ since $(\mathcal{M}_0, u_1)$ has modular objects (Δ_1, J_0). Since the cyclic and separating vector is uniquely determined (up to a selfadjoint element from the center $\mathbb{C}$ of $\mathcal{M}_0$) by the modular objects (Lemma A.2.1), there is a $c \in \mathbb{R}$ such that $\mathrm{W}^*u_1 = cu_2$. Hence,

$$\mathrm{U}^*u_0 = \mathrm{U}_2\mathrm{W}^*\mathrm{U}_1^*u_0 = \mathrm{U}_2\mathrm{W}^*u_1 = \mathrm{U}_2cu_2 = cu_0,$$

and $\mathrm{U} \in \mathcal{W}_1$. We have thus shown that $\mathcal{M}_1, \mathcal{M}_2 \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ are equivalent. □

Remark 6.4.3. Lemma 6.4.2 holds also in the type III_0 case with exactly the same proof.

6.5 The Generic Case

Beside the classification presented in §6.1 we can introduce a second, coarser classification for semifinite algebras and vectors diagonalizable with respect to the center. This allows to prove some results for the type II case which are analogous to those proved for the type I case in [Wol98] and [BW01].

In this section we discuss only semifinite factors $\mathcal{M}_0$. In particular, this implies that the central eigenvalues and multiplicities are numbers. The diagonalizability of an operator affiliated with $\mathcal{M}_0$ means that the operator possesses pure point spectrum, i.e. the closure of the set of all eigenvalues is the whole spectrum and the sum of the eigenprojections is the identity (cf. Remark 4.2.11).

Definition 6.5.1. Let $\mathcal{M}_0$ be a semifinite von Neumann factor with cyclic and separating vector u_0 and (Δ_0, J_0) be its modular objects. Suppose that $\Delta_0 = \mathrm{H}_0 \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0$ where $\mathrm{H}_0 \eta \mathcal{M}_0$ has pure point spectrum.

1. Let $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in NA(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ be a solution of the inverse problem and $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$ be the corresponding modular operator for $\mathcal{M}_0$. If $\mathrm{H}\eta\mathcal{M}_0$ has pure point spectrum we call $\mathcal{M}$ a *solution with pure point spectrum.* We denote the set of solutions with pure point spectrum by $\widehat{NA}(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$.

2. Two semifinite von Neumann factors
$$\mathcal{M}_1, \mathcal{M}_2 \in \widehat{NA}(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$$
are called *spectrum-equivalent*, $\mathcal{M}_1 \approx_s \mathcal{M}_2$, if the spectra of the corresponding operators H_1 and H_2 are equal up to a positive multiple, i.e. $\sigma(\mathrm{H}_1) = c\sigma(\mathrm{H}_2)$ for a $c \in \mathbb{R}_{>0}$, and the dimensions of the corresponding eigenprojections coincide.

3. The equivalence class of $\mathcal{M}_0$ with respect to "$\approx_s$" is denoted by $\widehat{NA}^1(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$.

Proposition 6.5.2. *The equivalence relation "$\approx_s$" is well-defined.*

Proof. To see the independence from the unitary U commuting with J_0 such that $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$ and $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$, let V be a second unitary commuting with J_0 such that $\mathrm{V}\mathcal{M}_0\mathrm{V}^* = \mathcal{M}$ and $\mathrm{V}^*\Delta_0\mathrm{V} = \mathrm{H}'\mathrm{J}_0(\mathrm{H}')^{-1}\mathrm{J}_0$. Then
$$\Delta_0 = \mathrm{U}\mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0\mathrm{U}^* = \mathrm{V}\mathrm{H}'\mathrm{J}_0(\mathrm{H}')^{-1}\mathrm{J}_0\mathrm{V}^*$$
which implies
$$\underbrace{\mathrm{V}^*\mathrm{U}\mathrm{H}\mathrm{U}^*\mathrm{V}}_{\in\mathcal{M}_0}\mathrm{J}_0\mathrm{V}^*\mathrm{U}\mathrm{H}^{-1}\mathrm{U}^*\mathrm{V}\mathrm{J}_0 = \mathrm{H}'\mathrm{J}_0(\mathrm{H}')^{-1}\mathrm{J}_0.$$
By Lemma A.1.1, it follows $\mathrm{V}^*\mathrm{U}\mathrm{H}\mathrm{U}^*\mathrm{V} = c\mathrm{H}'$ for a $c \in \mathbb{R}_{>0}$ since $\operatorname{ad}\mathrm{V}^*\mathrm{U} \in \operatorname{aut}\mathcal{M}_0$ and $\mathcal{M}_0$ is a factor. ☐

Remark 6.5.3.

1. In the type I case we have $\widehat{NA} = NA$ since every operator H which generates a modular operator possesses pure point spectrum (see Theorem 4.2.4).

2. In the type I case Definition 6.5.1 coincides with Definition 5.3.1. Indeed, let $\mathcal{M}_i \in NA(\mathcal{M}_0)$ $(i = 1, 2)$ be two type I factors such that $\mathcal{M}_1 \sim \mathcal{M}_2$. According to Lemma 6.1.1, there exists a unitary W such that $\mathrm{W}\mathrm{H}_2\mathrm{W}^* = c\mathrm{H}_1$, i.e. $\mathcal{M}_1 \approx_s \mathcal{M}_2$.

   On the other hand, if $\mathcal{M}_1 \approx_s \mathcal{M}_2$ then H_1 and H_2 have the same spectrum, i.e. they are unitarily equivalent in $L(\mathcal{H}_0) \simeq \mathcal{M}_0$. This implies that there is a unitary $\mathrm{W} \in \mathcal{M}_0$ such that $\mathrm{W}\mathrm{H}_2\mathrm{W}^* = c\mathrm{H}_1$, and Lemma 6.1.1 yields $\mathcal{M}_1 \sim \mathcal{M}_2$.

We define

$$\begin{aligned}\widehat{NA}^2(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) := \{\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in \widehat{NA}(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) |\\ \mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0 \quad \text{with } \mathrm{H}\eta\mathcal{M}_0,\\ \mathrm{H}^{-1} \text{ has the same eigenvalues}\\ \text{and dimensions of eigenspaces as } c\mathrm{H}_0 \text{ for a } c \in \mathbb{R}_{>0}.\}.\end{aligned} \tag{6.5.1}$$

Remark 6.5.4. The definition of $\widehat{NA^2}$ coincides with the one given for NA^2 in (5.3.1) if $\mathcal{M}_0$ is a type I factor. To see this, let $\mathcal{M}_0$ be a type I factor and $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in \widehat{NA^2}$. Hence, $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$ and H^{-1} has the same eigenvalues and dimensions of eigenspaces as $c\mathrm{H}_0$ for a $c \in \mathbb{R}_{>0}$. This implies the existence of a unitary $\mathrm{W} \in \mathcal{M}_0 \simeq L(\mathcal{H}_0)$ such that

$$\mathrm{H} = \mathrm{W}\mathrm{H}_0^{-1}\mathrm{W}^* = \underbrace{\mathrm{W}\mathrm{J}_0\mathrm{W}\mathrm{J}_0}_{=:\mathrm{V}}\mathrm{H}_0^{-1}\mathrm{W}^*\mathrm{J}_0\mathrm{W}^*\mathrm{J}_0$$

and $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{V}\Delta_0^{-1}\mathrm{V}^*$. Furthermore, since $(\mathcal{M}_0, \mathrm{U}^*u_1)$ has modular objects $(\mathrm{U}^*\Delta_0\mathrm{U}, \mathrm{J}_0)$, the operator Δ_0^{-1} is the modular operator corresponding to $\mathrm{V}^*\mathrm{U}^*u_0 =: u_1$. According to §5.3, there exists a unitary U_1 commuting with J_0 such that $\mathrm{U}_1 u_i = u_i$ $(i = 0, 1)$ and $\mathrm{U}_1^*\Delta_0\mathrm{U}_1 = \Delta_0^{-1}$. The unitary $\mathrm{K} := \mathrm{U}\mathrm{V}\mathrm{U}_1$ fulfils all prerequisites of (5.3.1) and we obtain $\mathcal{M} = \mathrm{K}\mathrm{U}_1\mathcal{M}_0\mathrm{U}_1^*\mathrm{K}^* \in NA^2(\Delta_0, \mathrm{J}_0, U_0; \mathcal{M}_0)$. The other inclusion is obvious.

In contrast to the type I case and the class NA^2 (see Theorem 5.3.11) we have the following proposition for the type II case.

Proposition 6.5.5. *The class $\widehat{NA^2}$ exists always (in analogy to the type I case) if $\mathcal{M}_0$ is finite. If $\mathcal{M}_0$ is infinite it does not always exist but sometimes.*

Proof.

1. Let $\mathcal{M}_0$ be finite. Since H_0 is assumed to possess pure point spectrum (to be diagonalizable with respect to the center in the terminology of §4.3) it has eigenvalues $(\mu_k)_{k\in K}$ with central multiplicities $(m_k)_{k\in K}$ such that $\sum_k m_k = 1$ and $\sum_k m_k\mu_k = 1$ (cf. Definition 4.3.1). Let $n_k := m_k\mu_k$ for every k. Then $(\mu_k^{-1})_{k\in K}$, $(n_k)_{k\in K}$ obviously fulfills the conditions of Lemma 6.3.6. This implies the existence of a solution such that the corresponding operator H has eigenvalues $(\mu_k^{-1})_{k\in K}$.

2. Let now $\mathcal{M}_0$ be infinite. Then H_0 has eigenvalues $(\mu_k)_{k\in K}$ with central multiplicities $(m_k)_{k\in K}$ such that $\sum_k m_k = \infty$ and $\sum_k m_k\mu_k = 1$. Assume further that $(\mu_k)_{k\in\mathbb{N}}$ is bounded, i. e. there is a $C > 0$ such that $\mu_k \leq C$ for all $k \in \mathbb{N}$ ($K = \mathbb{N}$ if $\mathcal{M}_0$ is infinite). Since in this case $\mu_k^{-1} \geq C^{-1}$ for every k, we have $\sum_k n_k\mu_k^{-1} \geq C^{-1}\sum_k n_k = \infty$ for all sequences $(n_k)_k$ with $\sum_k n_k = \infty$. This implies that there is no sequence $(n_k)_{k\in\mathbb{N}}$ with $\sum_k n_k = \infty$ such that the conditions of Theorem 6.3.8 are satisfied. Hence, there is no solution in $\widehat{NA^2}$.

   On the other hand, Example 6.5.6 shows that the class $\widehat{NA^2}$ may exist in the infinite case. □

Example 6.5.6. Let $\mathcal{M}_0$ be a type II_∞ factor. Consider the following sequences $(m_k)_{k\in\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$:

$$m_{2l+1} = 1/l, \quad m_{2l} = 1/l^3, \quad \mu_{2l} = l, \quad \text{and} \quad \mu_{2l+1} = 1/l \quad \text{for } l \in \mathbb{N}.$$

The sequences $(\mu_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ are the eigenvalues and multiplicities of a finite trace operator H_0 in $\mathcal{M}_0$ (Corollary 4.3.13). Let Δ_0 be the corresponding modular operator. The sequences $(\mu_k^{-1})_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ with

$$n_{2l+1} = 1/l^3 \quad \text{and} \quad n_{2l} = 1/l \quad \text{for } l \in \mathbb{N}$$

are also the eigenvalues and multiplicities of a finite trace operator H in $\mathcal{M}_0$ and the conditions of Theorem 6.3.8 are satisfied for them as well. Hence, they characterize a solution of the inverse problem which is in $\widehat{NA^2}$ since H and H_0^{-1} have the same eigenvalues (and infinite dimensional eigenspaces, see Remark 4.3.6).

Remark 6.5.7. Note that in Example 6.5.6 there is no automorphism $\alpha \in \mathrm{aut}(\mathcal{M}_0)$ such that $\mathrm{H}_0^{-1} = \alpha(\mathrm{H})$ since the multiplicities of the eigenvalues of H_0^{-1} and H do not coincide. Therefore, it can not be shown as in Remark 6.5.4 that Δ_0^{-1} is also a modular operator corresponding to a cyclic and separating vector. This example is hence no contradiction to Theorem 5.3.11.

To introduce the notion of generic spectrum of modular operators, we must first define some sets:

Definition 6.5.8. Let $D^K := \mathbb{R}^K$ be the set of all families in $\mathbb{R}$ indexed by a finite or countable set K ($K \in \{\{1, \dots, n\} | n \geq 3\} \cup \{\mathbb{N}\} =: M$), and let Ω^K be the following set:

$$\Omega^K := \{\bar{\alpha} = (\alpha_j)_{j\in K} \in D^K | \alpha_j \in \{-3, -2, -1, 0, 1, 2, 3\}, \sum_j \alpha_j = 0,$$
$$\text{at least two and at most six } \alpha_j \text{ are different from } 0\}.$$

With $\bar{\alpha} \in \Omega^K$ we define $D^K_{\bar{\alpha}} := \{\bar{x} \in D^K | \sum_{j\in K} \alpha_j x_j = 0\}$ and

$$D_{gen} := \bigcup_{K\in M} \bigcap_{\bar{\alpha}\in\Omega^K} (D^K \setminus D^K_{\bar{\alpha}}).$$

Furthermore, we set $\Gamma(\bar{x}) := \{x_j - x_s | j \neq s, j, s \in K\}$ for $\bar{x} \in D^K$.

Remark 6.5.9. 1. The significance of the sets defined in Definition 6.5.8 is the following. Let $\mathrm{H}_0 \eta \mathcal{M}_0$ be a positive invertible operator affiliated with a semifinite von Neumann factor $\mathcal{M}_0$. Assume that H_0 possesses pure point spectrum and let $(\mu_k)_{k\in K}$ be the family of eigenvalues of H_0, where we repeat each eigenvalue according to its multiplicity in the type I case, in the type II case every eigenvalue is counted only once. Then the family of logarithms $(\ln(\mu_k))_{k\in K}$ is an element of D^K. Hence, the set of possible spectra of positive invertible operators with pure point spectrum is a subset of $\bigcup_{K\in M} D^K$. The sets $D^K_{\bar{\alpha}}$ play the role of "small" exceptional sets.

2. The definition here differs from that given in [Wol98] and [BW01]. This is due to the fact that there it was assumed that the sequence of eigenvalues was normed according to the requirement that the largest eigenvalue equals 1. This is only possible in the type I case. Omitting this requirement, we must introduce the normalizing condition $\sum_j \alpha_j = 0$ and allow six α_j to be non-zero (cf. also Remark 6.5.11).

Definition 6.5.10. Let $\Delta_0 = \mathrm{H}_0 \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0$ be a modular operator for a semifinite von Neumann factor $\mathcal{M}_0$. Assume that $\mathrm{H}_0 \eta \mathcal{M}_0$ possesses pure point spectrum and let $(\mu_k)_{k\in K}$ be the family of eigenvalues of H_0, where we repeat each eigenvalue according to its multiplicity if $\mathcal{M}_0$ is a type I factor, if $\mathcal{M}_0$ is a type II factor every eigenvalue is counted only once. Then Δ_0 has *generic spectrum* if the family of the logarithms $(x_j)_{j\in K} = (\ln \mu_j)_{j\in K}$ is in D_{gen}.

Remark 6.5.11. 1. The notion of generic spectrum is independent of the operator H_0 generating the modular operator Δ_0: Since the decomposition of Δ_0 is unique up to a positive constant (see Lemma A.1.1) the logarithms of the eigenvalues can only differ by an additive constant $C \in \mathbb{R}$. Hence,

$$\sum_j \alpha_j(x_j + C) = \sum_j \alpha_j x_j + C \sum_j \alpha_j = \sum_j \alpha_j x_j$$

for $\bar{\alpha} = (\alpha_j) \in \Omega^K$.

2. It is a simple task to construct modular operators with generic spectrum as well as modular operators not with generic spectrum. Note, however, that "almost all" modular operator have generic spectrum, which legitimates the term "generic".

The remainder of this section is devoted to the proof of

Theorem 6.5.12. *Let $\mathcal{M}_0$ be a semifinite factor and (Δ_0, J_0) be the modular objects of $(\mathcal{M}_0, u_0)$. Moreover, Δ_0 is assumed to possess generic spectrum. Then*

$$\widehat{NA}(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) = \widehat{NA^1}(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0) \cup \widehat{NA^2}(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0),$$

where $\widehat{NA^2}$ can be empty if $\mathcal{M}_0$ is infinite.

Since Theorem 6.5.12 was already proved for the type I case in [Wol98] and [BW01] we only treat the type II case. For the proof we need the following lemmas.

Lemma 6.5.13. *Let $\bar{x} \in D_{gen}$. Then*

(i) $x_l - x_p = 0$ *if and only if* $l = p$.

(ii) $x_l - x_p + x_s - x_t = 0$ *if and only if* $l = p$ *and* $s = t$, *or* $l = t$ *and* $s = p$.

(iii) $x_l - x_p + x_s - x_t + x_r - x_o = 0$ *if and only if each index corresponding to an x with positive sign equals one index corresponding to an x with negative sign.*

Proof. This follows directly from the definition of D_{gen}. $\square$

Lemma 6.5.14. *Let $\bar{x} \in D_{gen}$ and $\bar{y} \in D^{K'}$ $(K' \in M)$.*

(i) The mapping defined by $z \in \Gamma(\bar{x}) \mapsto (j,s) \in K \times K$ for $z = x_j - x_s$ is well-defined.

(ii) Let $-\bar{x} := (-x_j)_{j \in K}$. Then $\Gamma(\bar{x}) = \Gamma(-\bar{x})$ and $\Gamma(\bar{x}) = \Gamma(\bar{x} + C)$ $(C \in \mathbb{R})$.

(iii) If $\Gamma(\bar{x}) = \Gamma(\bar{y})$ and $\{y_s\} \subset \{x_j + C\}$ for a $C \in \mathbb{R}$, then $\bar{y} = \bar{x} + C$.

Proof. (i) Let $\Gamma(\bar{x}) \ni z = x_j - x_s = x_k - x_t \neq 0$. Then $x_j - x_s - x_k + x_t = 0$, hence $j = k$ and $s = t$ (cf. Lemma 6.5.13).

(ii) is obvious.

(iii) Assume on the contrary that $\bar{y} \neq \bar{x} + C$. Then there exists an index $\alpha \in K$ such that $x_\alpha + C \neq y_j$ for all indices $j \in K'$. Let $\alpha \neq \beta \in K$ be an arbitrary index. Then $x_\alpha - x_\beta \in \Gamma(\bar{x}) = \Gamma(\bar{y})$ and there are indices $l, p \in K'$ such that $x_\alpha - x_\beta = y_l - y_p$. Since $\{y_s\} \subset \{x_j + C\}$ there are indices $m, n \in K$ such that $y_l = x_m + C$, $y_p = x_n + C$. This implies $x_\alpha - x_\beta = y_l - y_p = x_m - x_n$ and, by Lemma 6.5.13, $\alpha = m$. Hence $x_\alpha + C = y_l$, a contradiction. $\square$

Lemma 6.5.15. *Let $\bar{x} \in D_{gen}$ and $\bar{y} \in D^{K'}$ $(K' \in M)$ such that $\Gamma(\bar{x}) = \Gamma(\bar{y})$. Fix an index $k \in K'$.*

(i) Let $m, m' \in K'$ be two indices different from k such that $m \neq m'$. Then there are indices $n, n', o, o' \in K$ such that either

$$\begin{matrix} y_k - y_m = x_n - x_o & & y_k - y_m = x_n - x_o \\ \text{and} & \text{or} & \text{and} \\ y_k - y_{m'} = x_n - x_{o'} & & y_k - y_{m'} = x_{n'} - x_o \end{matrix} \tag{6.5.2}$$

holds.

(ii) If one of the two statements in (6.5.2) is true for one pair $m, m' \in K'$ with $m \neq m'$ the same statement is true for all such pairs.

(iii) Suppose that there exists an index $n \in K$ such that for every $m \in K'$ there is an index $o \in K$ such that $y_k - y_m = x_n - x_o$. Then the mapping $\sigma_k : K' \ni m \mapsto \sigma_k(m) = o \in K$ is well defined.

(iv) Suppose that there exists an index $o \in K$ such that for every $m \in K'$ there is an index $n \in K$ such that $y_k - y_m = x_n - x_o$. Then the mapping $\rho_k : K' \ni m \mapsto \sigma_k(m) = n \in K$ is well-defined.

(v) Suppose that one of the assumptions (iii) or (iv) holds. Then this assumption holds for all $k \in K'$.

Proof. (i) By assumption, for every $m \in K^{'}$ there are indices $n, o \in K$ such that $y_k - y_m = x_n - x_o$. Let now $k \neq m, m^{'} \in K^{'}$ two different indices. Then there are indices $n, n^{'}, o, o^{'} \in K$ such that

$$\begin{aligned} y_k - y_m &= x_n - x_o \\ y_k - y_{m'} &= x_{n'} - x_{o'}. \end{aligned}$$

This implies

$$\Gamma(\bar{y}) \ni y_{m'} - y_m = y_k - y_m - (y_k - y_{m'}) = x_n - x_o - (x_{n'} - x_{o'}).$$

Since $\Gamma(\bar{x}) = \Gamma(\bar{y})$ there are indices $r, s \in K$ such that $y_m - y_{m'} = x_r - x_s$ and

$$x_r - x_s - x_n + x_o + x_{n'} - x_{o'} = 0.$$

This equation is true if and only if $r = o^{'}$, $o = s$, $n^{'} = n$, or $r = n$, $o = o^{'}$, $n^{'} = s$ (see Lemma 6.5.13). Hence

$$\begin{array}{ccc} y_k - y_m = x_n - x_o & & y_k - y_m = x_n - x_o \\ \text{and} & \text{or} & \text{and} \\ y_k - y_{m'} = x_n - x_{o'} & & y_k - y_{m'} = x_{n'} - x_o. \end{array}$$

(ii) Assume that there are pairwise different indices $m, m^{'}, m^{''}$ such that indices $n, o, o^{'}, n^{''}$ exists with

$$\begin{aligned} y_k - y_m &= x_n - x_o, \\ y_k - y_{m'} &= x_n - x_{o'}, \\ y_k - y_{m''} &= x_{n''} - x_o. \end{aligned}$$

Then $y_{m''} - y_{m'} = x_n - x_{o'} - x_{n''} + x_o$ and $y_{m''} - y_{m'} \in \Gamma(\bar{y}) = \Gamma(\bar{x})$. Hence there are $i, j \in K$ such that $y_{m''} - y_{m'} = x_i - x_j$ and

$$x_i - x_j - x_n + x_{o'} + x_{n''} - x_o = 0.$$

According to Lemma 6.5.13 this implies $i = n$, $o^{'} = o$, $n^{''} = j$, or $i = o$, $o^{'} = j$, $n^{''} = n$. But in the first case we obtain $y_k - y_{m'} = x_n - x_o = y_k - y_m$, a contradiction to $m^{'} \neq m$ since $0 \notin \Gamma(\bar{x}) = \Gamma(\bar{y})$ (cf. Lemma 6.5.13.(i)). Similarly, we also obtain a contradiction in the second case. Hence, the two cases in (i) cannot hold simultaneously.

(iii) is obvious, cf. Lemma 6.5.14.(i).

(iv) is obvious, cf. Lemma 6.5.14.(i).

(v) Assume that there are indices k and $k^{'}$ such that the assumption of (iii) holds for k and that of (iv) holds for $k^{'}$. Then

$$\begin{aligned} y_k - y_m &= x_n - x_{\sigma_k(m)} \\ y_{k'} - y_m &= x_{\rho_{k'}(m)} - x_{o'} \end{aligned}$$

for all $m \in K' \setminus \{k, k'\}$. Then $y_k - y_{k'} = x_n - x_{\sigma_k(m)} - x_{\rho_{k'}(m)} + x_{o'}$ and $y_k - y_{k'} \in \Gamma(\bar{y}) = \Gamma(\bar{x})$. Since assumption (iii) holds for k we have $y_k - y_{k'} = x_n - x_{\sigma_k(k')}$ and

$$x_{\sigma_k(k')} - x_{\sigma_k(m)} - x_{\rho_{k'}(m)} + x_{o'} = 0.$$

This implies $\sigma_k(k') = \rho_{k'}(m)$ and $\sigma_k(m) = o'$. Analogously, $y_{k'} - y_k = x_{\rho_{k'}(k)} - x_{o'}$ implies $n = \rho_{k'}(m)$ and $\sigma_k(m) = \rho_{k'}(k)$. Hence $n = \rho_{k'}(m) = \sigma_k(k')$, a contradiction to $0 \neq y_k - y_{k'} = x_n - x_{\sigma_k(k')}$. □

Proof of Theorem 6.5.12. Note first that the definition of $\widehat{NA^1}$ implies that $\widehat{NA} \supset \widehat{NA^1}$, and Proposition 6.5.5 implies that $\widehat{NA} \supset \widehat{NA^2}$.

Let now $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in \widehat{NA}(\Delta_0, \mathrm{J}_0, u_0; \mathcal{M}_0)$ be a solution with pure point spectrum and $\Delta_1 := \mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}_1\mathrm{J}_0\mathrm{H}_1^{-1}\mathrm{J}_0$. Set $(x_j)_{j\in K} := (\ln \mu_j)_{j\in K}$ where $(\mu_j)_{j\in K}$ are the eigenvalues of H_0 (recall that $\Delta_0 = \mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0$). Then $\bar{x} = (x_j)_{j\in K} \in D_{gen}$ since Δ_0 has generic spectrum. Let further $(y_k)_{k\in K'} = (\ln \nu_k)_{k\in K'}$ be the logarithms of the eigenvalues $(\nu_k)_{k\in K'}$ of H_1. Then $\bar{y} = (y_k)_{k\in K'} \in D^{K'}$ and $\Gamma(\bar{x}) = \Gamma(\bar{y})$ because Δ_0 and Δ_1 are unitarily equivalent, i. e. they have the same eigenvalues. Now we can apply Lemma 6.5.15.

1. Lemma 6.5.15.(v) states that either the assumption of Lemma 6.5.15.(iii) is true for every $k \in K'$ or the assumption of Lemma 6.5.15.(iv). We assume that the assumption of Lemma 6.5.15.(iii) is true for every $k \in K'$. Then σ_k is independent of k. In fact, let $k, k', m \in K'$ be pairwise different. Then

$$\begin{aligned} y_k - y_m &= x_n - x_{\sigma_k(m)}, \\ y_{k'} - y_m &= x_{n'} - x_{\sigma_{k'}(m)}. \end{aligned}$$

This implies $x_n - x_{\sigma_k(m)} - x_{n'} + x_{\sigma_{k'}(m)} = y_k - y_{k'} = x_n - x_{\sigma_k(k')}$ according to Lemma 6.5.15.(iii) since $\Gamma(\bar{y}) = \Gamma(\bar{x})$, and

$$x_{\sigma_k(k')} - x_{\sigma_k(m)} - x_{n'} + x_{\sigma_{k'}(m)} = 0.$$

According to Lemma 6.5.13 this implies $\sigma_k(m) = \sigma_{k'}(m) =: \sigma(m)$ and $n' = \sigma(k')$ for all $m \in K'$. We conclude

$$y_{k'} - y_m = x_{\sigma(k')} - x_{\sigma(m)}$$

and

$$y_k = x_{\sigma(k)} + \underbrace{(y_m - x_{\sigma(m)})}_{=:C_1}$$

for all $k, m \in K'$ where C_1 is independent of both k and m. We have thus proved that $\{y_k\} \subset \{x_j + C_1\}$ and, according to Lemma 6.5.14, $\bar{y} = \bar{x} + C_1$.

2. Assume now on the contrary that there is no $C_1 \in \mathbb{R}$ such that $\bar{y} = \bar{x} + C_1$. According to the first part of the proof, there is an index $k \in K^{'}$ such that the assumption of Lemma 6.5.15.(iii) is false. Hence the assumption of Lemma 6.5.15.(iv) must be true for all $k \in K^{'}$. Now we can prove in full analogy with 1 that there is a $C_2 \in \mathbb{R}$ such that $\bar{y} = -\bar{x} + C_2$.

3. 1 and 2 prove that H_1 has the same spectrum as $e^{C_1}\mathrm{H}_0$ or $e^{C_2}\mathrm{H}_0^{-1}$. It hence corresponds to a solution in $\widehat{NA^1}$ or $\widehat{NA^2}$. $\square$

Chapter 7

Outlook

In this thesis a first big step was made towards a spectral theory of modular operators for von Neumann algebras (corresponding to cyclic and separating vectors). Furthermore, an example of its usefulness was given by applying it to the inverse problem. The theory is quite satisfactory for type I algebras since it is possible to characterize all modular operators by means of their spectral properties. Moreover, this leads to the complete classification of the solutions of the inverse problem. Similar results were also obtained for modular operators for type II algebras corresponding to cyclic and separating vectors which are diagonalizable with respect to the center. With this restriction, it is possible to characterize the modular operators for type II algebras by means of their spectral properties and to classify the solutions of the inverse problem as well. Furthermore, we gave first insights into the corresponding problems for type III_λ factors ($0 \leq \lambda < 1$).

A possible extension of the theory could concern modular operators corresponding to n. s. f. weights or cyclic and separating generalized vectors. However, since type I factors are isomorphic to $L(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ and the modular operators are hence decomposable into a tensor product of positive invertible operators affiliated with $L(\mathcal{H})$ this extension does not involve new methods, at least in the factor case. In this case, the spectral measure of the modular operator is the convolution of the spectral measures of the constituting operators. Hence, the modular operators for type I factors are exactly the positive invertible operators whose spectral measure is the convolution of two spectral measures.

For the non-factor case it could be useful to develop a center-valued spectral theory which we can only outline here. Let $\mathrm{M} \in \mathcal{M}$ be an element in a von Neumann algebra $\mathcal{M}$. Then we say $\mathrm{C} \in \mathcal{Z}(\mathcal{M})$ is in the *center-valued resolvent set* of M if $(\mathrm{C}-\mathrm{M})^{-1}$ exists and is in $\mathcal{M}$. Let $\rho_{\mathcal{Z}(\mathcal{M})}(\mathrm{M})$ be the set of all such elements. Then we call $\sigma_{\mathcal{Z}(\mathcal{M})}(\mathrm{M}) := \mathcal{Z}(\mathcal{M}) \setminus \rho_{\mathcal{Z}(\mathcal{M})}(\mathrm{M})$ the *center-valued spectrum* of M. In analogy to the usual spectral theory of normal operators on Hilbert spaces, the notions of point, continuous, and residual spectrum can also be defined. In this terminology Theorem 4.2.4 means that operators with finite trace have only central point spectrum (with the possible exception of functions being 0 on a non-null set which could belong to the continuous part of the central spectrum). Then the following questions arise: Is it possible to develop a functional calculus which is analogous to the classical one? Is it possible to define a spectral measure on the center, i. e. a mapping from the Borel-σ-algebra of the center into the set of projections of the algebra with the usual properties? Which topology must be used?

This theory has to be constructed in such a way that the classical spectral theory is the special case for the von Neumann factor $L(\mathcal{H})$.

The lack of a spectral theory adapted to operators which are elements of arbitrary von Neumann algebras becomes more crucial for factors which are not of type I. Up to now, it is not clear in general if and how the spectral properties of a modular operator can be derived from the spectral properties of its constituents in a simple way (cf. the discussion in §4.3.2). Progress in this subject would not only be interesting in its own right but would also help to classify the solutions of the inverse problems in the non-diagonalizable case. Furthermore, it might be useful for the conjugacy problem of the modular automorphism groups (cf. §6.3 and Remark 6.1.3). On the other hand, a progress in the latter could give a chance to classify the solutions of the inverse problems.

Even if all the above mentioned problems were solved for the semifinite case the type III problem would still remain complex. First of all, most of the theory developed in this thesis is only applicable to the III_λ case $(0 \leq \lambda < 1)$, not to the type III_1 case for which the discrete decomposition is in general no longer available. Moreover, although the spectral theory developed in §4.4 can essentially be reduced to the type II_∞ problem this proceeding does not provide a satisfactory result for the classification of the solutions of the inverse problems in the type III_λ case (cf. Lemma 6.4.2). Furthermore, the relation between the inverse problem and the conjugacy problem of the modular automorphism groups does no longer exist. These observations suggest the conjecture that we must modify the equivalence relation for type III factors or, possibly, apply completely different techniques. A possible candidate might be the continuous decomposition of type III factors which gives a decomposition of type III factors into a crossed product of a type II_∞ factor and a continuous group action. Moreover, this continuous decomposition is even more adapted to modular theory (see e. g. [Yam92]).

Appendix A

Some Auxiliary Results

In this appendix we prove some lemmas which were used in the main part of this thesis. Because they might be known (Lemma A.1.1, Lemma A.2.1, Lemma A.4.1) or their proofs are lengthy(Lemma A.3.1, Lemma A.3.5) they are postponed into the appendix.

A.1 Product of Commuting Operators

The first lemma states that a product of commuting operators determines uniquely its factors up to a central element:

Lemma A.1.1. *Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Let further $\Delta = \mathrm{HH}' = \mathrm{GG}'$ a closed invertible operator on $\mathcal{H}$ where $\mathrm{H}, \mathrm{G} \eta \mathcal{M}$ and $\mathrm{H}', \mathrm{G}' \eta \mathcal{M}'$. Then there exists an invertible operator $\mathrm{C} \eta \mathcal{Z}(\mathcal{M})$ such that $\mathrm{H} = \mathrm{CG}$ and $\mathrm{H}' = \mathrm{C}^{-1}\mathrm{G}'$.*

Proof. Assume first that H, H', G, and G' are positive. Then Δ is positive as well. We now examine the unitary group $\Delta^{it} = \mathrm{H}^{it}(\mathrm{H}')^{it} = \mathrm{G}^{it}(\mathrm{G}')^{it}$ (the splitting is possible since H commutes with H', and G commutes with G'). This equality implies $\mathrm{G}^{-it}\mathrm{H}^{it} = (\mathrm{G}')^{it}(\mathrm{H}')^{-it} \in \mathcal{Z}(\mathcal{M})$ for all $t \in \mathbb{R}$. Since G^{-it} and H^{it} also commute for all $t \in \mathbb{R}$ (which is shown by a simple computation) $\mathrm{G}^{-it}\mathrm{H}^{it}$ is a unitary group in $\mathcal{Z}(\mathcal{M})$. Thus,

$$\mathrm{G}^{-it}\mathrm{H}^{it} = \mathrm{C}^{it} \text{ with } 0 < \mathrm{C} \eta \mathcal{Z}(\mathcal{M})$$

which, by uniqueness of the generator of a group, implies the assertion.

Let now $\mathrm{H} = \mathrm{UA}$, $\mathrm{H}' = \mathrm{U}'\mathrm{A}'$, $\mathrm{G} = \mathrm{VB}$, and $\mathrm{G}' = \mathrm{V}'\mathrm{B}'$ be the polar decompositions of H, H', G, and G' with $\mathrm{U}, \mathrm{V} \in \mathcal{U}(\mathcal{M})$, $\mathrm{U}', \mathrm{V}' \in \mathcal{U}(\mathcal{M}')$, $0 < \mathrm{A}, \mathrm{B} \eta \mathcal{M}$, and $0 < \mathrm{A}', \mathrm{B}' \eta \mathcal{M}'$. Then $\Delta = \mathrm{UAU}'\mathrm{A}' = \mathrm{VBV}'\mathrm{B}'$ are two polar decompositions of Δ. By uniqueness of polar decomposition we obtain $\mathrm{UU}' = \mathrm{VV}'$ and $\mathrm{AA}' = \mathrm{BB}'$. The first part now implies the assertion. ☐

A.2 Uniqueness of Cyclic and Separating Vectors

In this section we show that a cyclic and separating vector is uniquely determined (up to a “real” element in the center) by the modular objects corresponding to it.

Lemma A.2.1. *Let $u \in \mathcal{H}$ be a cyclic and separating vector for a von Neumann algebra $\mathcal{M}$ acting on $\mathcal{H}$ and (Δ, J) be the corresponding modular objects. Suppose that $v \in \mathcal{H}$ is another cyclic and separating vector with modular objects (Δ, J). Then it exists a selfadjoint invertible operator $\mathrm{C}\eta\mathcal{Z}(\mathcal{M})$ such that $u \in \mathcal{D}(\mathrm{C})$ and $v = \mathrm{C}u$.*

Proof. Let $P_u^\natural$ and $P_v^\natural$ be the natural cones of u and v, respectively. Since u and v have the same modular conjugation J there is a unitary $\mathrm{U} \in \mathcal{U}(\mathcal{M}')$ commuting with J such that $\mathrm{U}P_u^\natural = P_v^\natural$. In fact, if we take $\pi = \mathrm{id}$, Theorem 2.1.10 yields the existence of a unitary U commuting with J such that $\mathrm{U}P_u^\natural = P_v^\natural$ and $\mathrm{id} = \mathrm{ad}\,\mathrm{U}$. The latter implies $\mathrm{U} \in \mathcal{U}(\mathcal{M}')$. Since $\mathrm{U} \in \mathcal{M}'$ commutes with J we even have $\mathrm{U} \in \mathcal{Z}(\mathcal{M})$. Moreover, U is selfadjoint since Tomita's Theorem (Theorem 2.1.3) implies $\mathrm{JUJ} = \mathrm{U}^*$.

Since the modular automorphism groups of u and v coincide there is a positive invertible operator $\mathrm{A}\eta\mathcal{Z}(\mathcal{M})$ such that $v \in \mathcal{D}(\mathrm{A})$ and $\mathrm{A}v$ generate the same vector state ω as u (see [Str81, Corollary 4.11]). Then $\mathrm{A}v = \mathrm{A}^{1/2}\mathrm{J}\mathrm{A}^{1/2}\mathrm{J}v \in P_v^\natural$ is the unique vector in the natural cone of v generating ω. Moreover, $\mathrm{U}u \in P_v^\natural$ generates ω, hence $\mathrm{A}v = \mathrm{U}u$. Setting now $\mathrm{C} := \mathrm{A}^{-1}\mathrm{U}\eta\mathcal{Z}(\mathcal{M})$ we get an invertible and selfadjoint operator since A and U commute. Finally, $v = \mathrm{C}u$. □

Remark A.2.2. Suppose that u is a cyclic and separating vector for a von Neumann algebra $\mathcal{M}$ and C is a selfadjoint invertible operator affiliated with the center of $\mathcal{M}$ such that $u \in \mathcal{D}(\mathrm{C})$. Then a simple calculation shows that $\mathrm{C}u$ is also a cyclic and separating vector for $\mathcal{M}$ with the same modular objects as u (cf. also §3.2 where similar calculations were used).

A.3 Two Lemmas

For some constructions in Chapter 5 we need the following two lemmas which are concerned with the the existence of a special conjugation (Lemma A.3.1) and a special unitary (Lemma A.3.5).

Lemma A.3.1. *Let Δ be a positive invertible operator on a Hilbert space $\mathcal{H}$ and J be a conjugation on the same Hilbert space such that*

$$\mathrm{J}\Delta\mathrm{J} = \Delta^{-1}.$$

Suppose that there are two vectors $v_1, v_2 \in \mathcal{H}$ such that

$$\Delta\mathrm{A}v_i = \mathrm{A}v_i \quad \textit{and} \quad \mathrm{J}\mathrm{A}v_i = \mathrm{A}^*v_i \quad (i = 1, 2)$$

for all $\mathrm{A} \in \mathcal{A}$ where $\mathcal{A}$ is a von Neumann algebra. Then there exists a conjugation L on $\mathcal{H}$ such that

$$\mathrm{L}\Delta\mathrm{L} = \Delta, \quad \mathrm{IJI} = \mathrm{J} \;\textit{and}\; \mathrm{LA}v_i = \mathrm{A}^*v_i \quad (i = 1, 2).$$

To prove this lemma we need some preparatory results.

Proposition A.3.2. *Let Δ be a selfadjoint operator on a Hilbert space $\mathcal{H}$. Then there is a conjugation* K *such that Δ is* K*-real, i. e.*

$$\mathrm{K}\Delta\mathrm{K} = \Delta$$

(cf. [Wei76, p.223, ex. 8.1]).

Proof. Since Δ is selfadjoint there exists a measure space $(\Omega, \mathfrak{A}, \mu)$, a unitary $\mathrm{U} : \mathcal{H} \to L_2(\mu)$, and a real valued measurable function g such that

$$(\mathrm{U}\Delta\mathrm{U}^* f)(t) = g(t)f(t) \ \mu\text{-almost everywhere on } \Omega$$

holds for every $f \in L_2(\mu)$ with $\mathrm{U}^* f \in \mathcal{D}(\Delta)$. Define now

$$(\tilde{\mathrm{K}} f)(t) = \overline{f}(t).$$

Then $\mathrm{K} := \mathrm{U}^* \tilde{\mathrm{K}} \mathrm{U}$ is the claimed conjugation. □

Proposition A.3.3. *Let Δ be a* K*-real selfadjoint operator on $\mathcal{H}$ where* K *is a conjugation on $\mathcal{H}$.*

1. *$\mathcal{K} := \{u \in \mathcal{H} | u = \mathrm{K}u\}$ is a real subspace of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{K} + i\mathcal{K}$ and $\langle \cdot | \cdot \rangle_{\mathcal{K}} := \langle \cdot | \cdot \rangle_{|\mathcal{K}}$ is a real scalar product on $\mathcal{K}$.*

2. *$\Delta_{\mathcal{K}} := \Delta_{|\mathcal{K}}$ is a selfadjoint operator on the real vector space $\mathcal{K}$.*

Proof. 1. Let $v \in \mathcal{H}$. Then

$$v = \underbrace{\frac{v + \mathrm{K}v}{2}}_{\in \mathcal{K}} + i \underbrace{\frac{v - \mathrm{K}v}{2i}}_{\in \mathcal{K}}.$$

If $v, w \in \mathcal{K}$ then

$$\langle v | w \rangle = \langle \mathrm{K}v | \mathrm{K}w \rangle = \langle w | v \rangle.$$

2. Let $u \in \mathcal{K} \cap \mathcal{D}(\Delta)$. Then

$$\mathrm{K}\Delta u = \Delta \mathrm{K} u = \Delta u.$$

This implies that $\Delta(\mathcal{K} \cap \mathcal{D}(\Delta)) \subset \mathcal{K}$. The selfadjointness of $\Delta_{\mathcal{K}}$ follows from standard calculations. □

Proposition A.3.4. *Let $\Delta, \mathrm{J}, \mathcal{H}, \mathcal{K}$ be as in Proposition A.3.3. Then there is an orthonormal base (ONB for short) $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(\Delta)$ for the real vector space $\mathcal{K}$. Moreover, $\{u_k\}$ is also an ONB for the complex vector space $\mathcal{H}$ such that $\langle \Delta u_k | u_l \rangle \in \mathbb{R}$ for $k, l \in \mathbb{N}$.*

Proof. This follows immediately from Proposition A.3.3. □

Proof of Lemma A.3.1. 1. Let E be the spectral measure of Δ. Then we decompose $\mathcal{H}$ into the direct sum of the following three orthogonal subspaces:

$$\mathcal{H} = \mathcal{K}_{-1} \oplus \mathcal{K}_0 \oplus \mathcal{K}_1$$

where $\mathcal{K}_{-1} := \mathrm{E}(\{\lambda < 1\})$, $\mathcal{K}_0 := \mathrm{E}(\{\lambda = 1\})$, $\mathcal{K}_1 := \mathrm{E}(\{\lambda > 1\})$. Then $\Delta(\mathcal{K}_j \cap \mathcal{D}(\Delta)) \subset \mathcal{K}_j$ $(j = -1, 0, 1)$. $\mathrm{J}\Delta\mathrm{J} = \Delta^{-1}$ implies $\mathrm{J}\mathcal{K}_j \subset \mathcal{K}_{-j}$ and, since $\mathrm{J}^2 = \mathrm{I}$, equality holds as well.

2. In $\mathcal{K}_0$ we set $\mathrm{L}_0 := \mathrm{J}_{|\mathcal{K}_0}$. Then

$$\mathrm{L}_0 \Delta_{|\mathcal{K}_0} \mathrm{L}_0 = \mathrm{I}_{|\mathcal{K}_0} = \Delta_{|\mathcal{K}_0} \tag{A.3.1}$$

and

$$\mathrm{L}_0 \mathrm{J}_{|\mathcal{K}_0} \mathrm{L}_0 = \mathrm{J}_{|\mathcal{K}_0}. \tag{A.3.2}$$

Since $\mathrm{A}v_i \in \mathcal{K}_0$ for $\mathrm{A} \in \mathcal{A}$ we obtain

$$\mathrm{L}_0 \mathrm{A} v_i = \mathrm{J}\mathrm{A} v_i = \mathrm{A}^* v_i \quad (i = 1, 2). \tag{A.3.3}$$

3. In $\mathcal{K}_1$ we choose an ONB $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(\Delta)$ according to Proposition A.3.4. Setting $u_{-k} := \mathrm{J}u_k \in \mathcal{K}_{-1}$ $(k \in \mathbb{N})$ we conclude that $\{u_{-k}\} \subset \mathcal{D}(\Delta)$ is an ONB in $\mathcal{K}_{-1}$. We define a conjugation L_j on $\mathcal{K}_j$ $(j = -1, 1)$ by

$$\mathrm{L}_j \alpha u_{jk} := \overline{\alpha} u_{jk} \qquad \text{for } \alpha \in \mathbb{C} \text{ and } k \in \mathbb{N}. \tag{A.3.4}$$

Let $v_j = \sum_l \alpha_{jl} u_{jl} \in \mathcal{K}_j$. Then

$$\begin{aligned}
\big\langle \mathrm{L}_j \Delta_{|\mathcal{K}_j} \mathrm{L}_j u_{jk} \big| \sum_l \alpha_{jl} u_{jl} \big\rangle &= \big\langle \mathrm{L}_j \sum_l \alpha_{jl} u_{jl} \big| \Delta_{|\mathcal{K}_j} \mathrm{L}_j u_{jk} \big\rangle \\
&= \sum_l \overline{\alpha_{jl}} \quad \underbrace{\big\langle u_{jl} \big| \Delta_{|\mathcal{K}_j} u_{jk} \big\rangle}_{\in \mathbb{R} \text{ (see Proposition A.3.4)}} \\
&= \sum_l \overline{\alpha_{jl}} \big\langle \Delta_{|\mathcal{K}_j} u_{jk} \big| u_{jl} \big\rangle \\
&= \big\langle \Delta_{|\mathcal{K}_j} u_{jk} \big| v_j \big\rangle.
\end{aligned}$$

By linear continuation, this yields

$$\mathrm{L}_j \Delta_{|\mathcal{K}_j} \mathrm{L}_j = \Delta_{|\mathcal{K}_j} \tag{A.3.5}$$

for $j = -1, 1$.

4. Setting $\mathrm{L} := \mathrm{L}_{-1} \oplus \mathrm{L}_0 \oplus \mathrm{L}_1$ we deduce

$$\mathrm{L}\Delta\mathrm{L} = \Delta$$

from (A.3.1) and (A.3.5). If $u = u_0 + \sum_{j=-1,1;k} \alpha_{jk} u_{jk} \in \mathcal{H}$ where $u_0 \in \mathcal{K}_0$, (A.3.2) and (A.3.4) imply

$$\begin{aligned}
\mathrm{L}\mathrm{J}\mathrm{L}u &= \mathrm{J}u_0 + \sum_{j,k} \overline{\alpha_{jk}} \mathrm{L}_j \mathrm{J} \mathrm{L}_j u_{jk} \\
&= \mathrm{J}u_0 + \sum_{j,k} \overline{\alpha_{jk}} u_{-jk} \\
&= \mathrm{J}u_0 + \mathrm{J}\sum_{j,k} \alpha_{jk} u_{jk} = \mathrm{J}u.
\end{aligned}$$

By (A.3.3), we conclude

$$\mathrm{LA}v_i = \mathrm{JA}v_i = \mathrm{A}^*v_i \quad (i = 1, 2)$$

for $\mathrm{A} \in \mathcal{A}$ and the proof is complete. □

Lemma A.3.5. *Let Δ_1, Δ_2 be two unitarily equivalent, positive invertible operators acting on a Hilbert space $\mathcal{H}$ and J be a conjugation on the same Hilbert space such that*

$$\mathrm{J}\Delta_1\mathrm{J} = \Delta_1^{-1} \quad \text{and} \quad \mathrm{J}\Delta_2\mathrm{J} = \Delta_2^{-1}.$$

Then there is a unitary U commuting with J such that $\Delta_2 = \mathrm{U}\Delta_1\mathrm{U}^*$.

Proof. Let E^i be the spectral measure of Δ_i $(i = 1, 2)$ and $\mathrm{V} \in \mathcal{U}(\mathcal{H})$ be a unitary such that $\Delta_2 = \mathrm{V}\Delta_1\mathrm{V}^*$. As in the proof of Lemma A.3.1 we can decompose $\mathcal{H}$ into

$$\mathcal{H} = \mathcal{K}_{-1}^1 \oplus \mathcal{K}_0^1 \oplus \mathcal{K}_1^1 \quad \text{and} \quad \mathcal{H} = \mathcal{K}_{-1}^2 \oplus \mathcal{K}_0^2 \oplus \mathcal{K}_1^2$$

where $\mathcal{K}_{-1}^i := \mathrm{E}^i(\{\lambda < 1\})$, $\mathcal{K}_0^i := \mathrm{E}^i(\{\lambda = 1\})$, $\mathcal{K}_1^i := \mathrm{E}^i(\{\lambda > 1\})$. Then $\mathrm{J}\mathcal{K}_j^i = \mathcal{K}_{-j}^i$ and $\mathrm{V}\mathcal{K}_j^1\mathrm{V}^* = \mathcal{K}_j^2$ $(i = 1, 2$ and $j = -1, 0, 1)$. This implies that we can define unitaries $\mathrm{W}_1\mathcal{K}_1^1 \to \mathcal{K}_1^2$ and $\mathrm{W}_{-1} : \mathcal{K}_{-1}^1 \to \mathcal{K}_{-1}^2$ by

$$\mathrm{W}_1 := \mathrm{V}_{|\mathcal{K}_1^1} \quad \text{and} \quad \mathrm{W}_{-1} = \mathrm{JW}_1\mathrm{J}.$$

Consider now the real vector spaces $\tilde{\mathcal{K}}_0^i := \{u \in \mathcal{K}_0^i | \mathrm{J}_0 u = u\}$ $(i = 1, 2)$. Since $\mathcal{K}_0^1$ and $\mathcal{K}_0^2$ are unitarily equivalent there is an orthogonal mapping $\mathrm{O} : \tilde{\mathcal{K}}_0^1 \to \tilde{\mathcal{K}}_0^2$. Defining $\mathrm{W}_0(u + iv) := \mathrm{O}u + i\mathrm{O}v$ we get a unitary W_0 from $\mathcal{K}_0^1$ onto $\mathcal{K}_0^2$ which commutes with J.

Finally, we define a unitary U on $\mathcal{H}$ by

$$\mathrm{U} := \mathrm{W}_{-1} \oplus \mathrm{W}_0 \oplus \mathrm{W}_1$$

which commutes with J. Moreover, the above calculations imply

$$\begin{aligned}
\mathrm{U}\Delta_1\mathrm{U}^* &= \mathrm{U}\left(\int_{\{\lambda<1\}} \lambda d\mathrm{E}_\lambda^1 + (\Delta_1)_{|\mathcal{K}_0^1} + \int_{\{\lambda>1\}} \lambda d\mathrm{E}_\lambda^1\right)\mathrm{U}^* \\
&= \mathrm{W}_{-1}\left(\int_{\{\lambda<1\}} \lambda d\mathrm{E}_\lambda^1\right)\mathrm{W}_{-1} + \mathrm{W}_0\mathrm{I}_{\mathcal{K}_0^1}\mathrm{W}_0^* + \mathrm{W}_1\left(\int_{\{\lambda>1\}} \lambda d\mathrm{E}_\lambda^1\right)\mathrm{W}_1^* \\
&= \mathrm{JW}_1\left(\int_{\{\lambda>1\}} \lambda d\mathrm{E}_\lambda^1\right)\mathrm{W}_1\mathrm{J} + \mathrm{I}_{\mathcal{K}_0^2} + \int_{\{\lambda>1\}} \lambda d\mathrm{E}_\lambda^2 \\
&= \mathrm{J}\left(\int_{\{\lambda>1\}} \lambda d\mathrm{E}_\lambda^2\right)\mathrm{J} + \mathrm{I}_{\mathcal{K}_0^2}^* + \int_{\{\lambda>1\}} \lambda d\mathrm{E}_\lambda^2 = \Delta_2.
\end{aligned}$$

□

A.4 Tensor Product of Type III_λ Factors

In this section we consider the type of the tensor product of two III_λ factors. Note first that it is of type III according to Table 2.1.

Lemma A.4.1. *The tensor product of a type III_{λ_1} factor with a type III_{λ_2} factor ($0 < \lambda_1, \lambda_2 < 1$) is of type III_λ where*

$$\lambda = \begin{cases} \lambda_1^{1/m} & \text{if } \ln(\lambda_2)/\ln(\lambda_1) = n/m \text{ for } n, m \in \mathbb{Z} \text{ and } (n,m) = 1 \\ 1 & \text{otherwise} \end{cases} \tag{A.4.1}$$

where (n,m) denotes the largest common divisor of n and m.

Proof. Let $\mathcal{M}_i$ be a type III_{λ_i} factor ($0 < \lambda_i < 1$, $i = 1,2$). This implies the existence of a generalized trace τ_i such that

$$\begin{gathered} \sigma_{\tau_i}^{T_i} = \mathrm{id} \ \text{ for } T_i := -2\pi/\ln(\lambda_i), \\ \tau_i(\mathrm{I}) = \infty, \text{ and} \\ \sigma(\Delta_{\tau_i}) \cap \mathbb{R}_*^+ = \{\lambda_i^z | z \in \mathbb{Z}\} \end{gathered}$$

(cf. §2.3 and Proposition 3.4.3). The n. s. f. weight $\tau := \tau_1 \otimes \tau_2$ is a weight on $\mathcal{M}_1 \otimes \mathcal{M}_2$ such that

$$\sigma(\Delta_\tau) = \sigma(\Delta_{\tau_1} \otimes \Delta_{\tau_2}) = \overline{\{\lambda_1^{z_1}\lambda_2^{z_2} | z_1, z_2 \in \mathbb{Z}\}}.$$

If $\ln(\lambda_2)/\ln(\lambda_1) = n/m$ with $n, m \in \mathbb{Z}$ and $(n,m) = 1$ then

$$\sigma(\Delta_\tau) = \{\lambda_1^{z/m} | z \in \mathbb{Z}\} \cup \{0\}$$

otherwise

$$\sigma(\Delta_\tau) = [0, \infty)\,.$$

Since $\mathcal{Z}((\mathcal{M}_1 \otimes \mathcal{M}_2)^{\tau_1 \otimes \tau_2}) = \mathcal{Z}(\mathcal{M}_1^{\tau_1}) \otimes \mathcal{Z}(\mathcal{M}_2^{\tau_2})$ and $\mathcal{M}_1^{\tau_1}$ and $\mathcal{M}_2^{\tau_2}$ are factors $(\mathcal{M}_1 \otimes \mathcal{M}_2)^{\tau_1 \otimes \tau_2}$ is also a factor and, according to [Str81, Theorem 28.3], $S(\mathcal{M}_1 \otimes \mathcal{M}_2) \cap \mathbb{R}_*^+ = \sigma(\Delta_\tau)$. This proves that $\mathcal{M}_1 \otimes \mathcal{M}_2$ is of type III_λ where λ is defined in (A.4.1) (cf. Definition 2.3.2). □

Appendix B

Proof of Theorem 5.2.9

Here we give the proof of Theorem 5.2.9. Adopt the notations of §5.2.

Theorem. *A von Neumann factor* $\mathcal{M}$ *belongs to* $NF(\Delta_0, u_0; \mathcal{M}_0)$ *if and only if there is a unitary operator* U *such that*

(i) $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$,

(ii) there are unitaries $\mathrm{K}, \mathrm{Y}_1, \mathrm{Y}_2$ *on* $\mathcal{H} \otimes \mathcal{H}_\rho$ *with* $\mathrm{K} \in \{\Delta_0 \otimes \Delta_\rho\}'$ *and*

$$\mathrm{U} \otimes \mathrm{I}_\rho = \mathrm{K} \cdot \mathrm{Y}_1 \cdot \mathrm{Y}_2, \quad \mathrm{Y}_1 \in \mathcal{M}_0 \otimes \mathcal{F}_\rho, \quad \mathrm{Y}_2 \in \mathcal{M}_0' \otimes \mathcal{F}_\rho',$$

(iii) $\omega_0 \otimes \rho_1(\cdot) = c(\omega_0 \otimes \rho_1)(\mathrm{K} \cdot \mathrm{K}^*)$ *as weights on* $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ *and* $\mathcal{M}_0' \otimes \mathcal{F}_\rho$ *with a* $c > 0$ *(and* $\omega_0 \otimes \rho_1)(\mathrm{K} \cdot \mathrm{K}^*) := (\omega_\mathrm{U} \otimes \rho_1)(\mathrm{Y}_2^*\mathrm{Y}_1^* \cdot \mathrm{Y}_1\mathrm{Y}_2)$*)*.

Proof of Theorem 5.2.9. 1. Recall first that $\mathcal{M} \in NF(\Delta_0, u_0; \mathcal{M}_0)$ if and only if there is a unitary U such that $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^*$, U^*u_0 is a cyclic and separating vector for $\mathcal{M}_0$ and $\Delta := \mathrm{U}^*\Delta_0\mathrm{U}$ is the modular operator for $(\mathcal{M}_0, \mathrm{U}^*u_0)$ (see Remark 5.2.5). We have therefore to prove that U^*u_0 is a cyclic and separating vector for $\mathcal{M}_0$ and Δ is the modular operator for $(\mathcal{M}_0, \mathrm{U}^*u_0)$ if and only if $\mathrm{U} \otimes \mathrm{I}_\rho$ has the properties described in the theorem.

2. Suppose that $\Delta := \mathrm{U}^*\Delta_0\mathrm{U}$ is the modular operator for $(\mathcal{M}_0, \mathrm{U}^*u_0)$. Then, by Theorem 2.1.15, there is a cocycle $\mathrm{U}_t \in \mathcal{M}_0$ for $\mathrm{ad}\,\Delta_0^{it}$ such that

$$\mathrm{ad}\,\Delta^{it} = \mathrm{ad}(\mathrm{U}_t\Delta_0^{it}) \quad t \in \mathbb{R}.$$

This relation implies that

$$\mathrm{U}^*\Delta_0^{it}\mathrm{U} = \Delta^{it} = \mathrm{W}_t\mathrm{U}_t\Delta_0^{it} \quad t \in \mathbb{R} \tag{B.0.1}$$

where the unitaries $\mathrm{W}_t \in \mathcal{M}_0'$ also satisfy the cocycle condition

$$\mathrm{W}_{t+s} = \mathrm{W}_t(\Delta_0^{it}\mathrm{W}_s\Delta_0^{-it}) \quad (t, s \in \mathbb{R}).$$

Now we define $\mathrm{L}_t := \mathrm{J}_0\mathrm{W}_t\mathrm{J}_0$. Thus $\mathrm{L}_t \in \mathcal{M}_0$ and L_t is a cocycle for $\mathrm{ad}\,\Delta_0^{it}$.

We can connect these cocycles $\mathrm{U}_t, \mathrm{L}_t$ with elements V, L from $\mathcal{M}_0 \otimes \mathcal{F}_\infty$. Now the proof of the converse of the Connes Cocycle Theorem (cf. [Str81, 5.4]) implies that there are unitaries $\mathrm{V}, \mathrm{L} \in \mathcal{M}_0 \otimes \mathcal{F}_\infty$ such that

$$\mathrm{U}_t \otimes \mathrm{I} = \mathrm{V}(\mathrm{ad}\,\Delta_0^{it} \otimes \sigma_t^\rho)(\mathrm{V}^*) \quad \text{on } \mathcal{H} \otimes L_2(\mathbb{R}, \lambda).$$
$$\mathrm{L}_t \otimes \mathrm{I} = \mathrm{L}(\mathrm{ad}\,\Delta_0^{it} \otimes \sigma_t^\rho)(\mathrm{L}^*) \quad \text{on } \mathcal{H} \otimes L_2(\mathbb{R}, \lambda).$$

We denote the corresponding unitaries from $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ on $\mathcal{H} \otimes \mathcal{H}_\rho$ by $\mathrm{V}_\rho, \mathrm{L}_\rho$. Then we define

$$\mathrm{W}_\rho := \tilde{\mathrm{J}}\mathrm{L}_\rho\tilde{\mathrm{J}} \quad \text{with } \tilde{\mathrm{J}} := \mathrm{J}_0 \otimes \mathrm{J}_\rho.$$

Thus $\mathrm{W}_\rho \in (\mathcal{M}_0 \otimes \mathcal{F}_\rho)' = \mathcal{M}_0' \otimes \mathcal{F}_\rho'$. Furthermore, we get

$$\mathrm{U}_t \otimes \mathrm{I}_\rho = \mathrm{V}_\rho \tilde{\sigma}_t(\mathrm{V}_\rho^*), \quad \mathrm{W}_t \otimes \mathrm{I}_\rho = \mathrm{W}_\rho \tilde{\sigma}_t(\mathrm{W}_\rho^*). \tag{B.0.2}$$

Next, we define a unitary K by $\mathrm{K} := (\mathrm{U} \otimes \mathrm{I}_\rho)\mathrm{W}_\rho \mathrm{V}_\rho$. Now (B.0.1) implies the following relation on the tensor product $\mathcal{H} \otimes \mathcal{H}_\rho$,

$$(\mathrm{U}^* \otimes \mathrm{I}_\rho)(\Delta_0^{it} \otimes \Delta_\rho^{it})(\mathrm{U} \otimes \mathrm{I}_\rho)(\Delta_0^{-it} \otimes \Delta_\rho^{-it}) = (\mathrm{W}_t \otimes \mathrm{I}_\rho)(\mathrm{U}_t \otimes \mathrm{I}_\rho). \tag{B.0.3}$$

Inserting $\mathrm{U} \otimes \mathrm{I}_\rho = \mathrm{K}\mathrm{V}_\rho^*\mathrm{W}_\rho^*$ and using (B.0.2) we obtain

$$\mathrm{W}_\rho \mathrm{V}_\rho \mathrm{K}^* \tilde{\sigma}_t(\mathrm{K}\mathrm{W}_\rho^*\mathrm{V}_\rho^*) = \mathrm{W}_\rho \tilde{\sigma}_t(\mathrm{W}_\rho^*)\mathrm{V}_\rho \tilde{\sigma}_t(\mathrm{V}_\rho^*).$$

Since $\tilde{\sigma}_t$ is an automorphism for $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ as well as for $\mathcal{M}_0' \otimes \mathcal{F}_\rho'$ and since W_ρ and V_ρ commute we obtain

$$\mathrm{W}_\rho \mathrm{V}_\rho \mathrm{K}^* \tilde{\sigma}_t(\mathrm{K})\tilde{\sigma}_t(\mathrm{W}_\rho^*\mathrm{V}_\rho^*) = \mathrm{W}_\rho \mathrm{V}_\rho \tilde{\sigma}_t(\mathrm{W}_\rho^*\mathrm{V}_\rho^*).$$

Furthermore,

$$\mathrm{K}^*\tilde{\sigma}_t(\mathrm{K}) = \mathrm{I} \text{ or } \mathrm{K} = \tilde{\sigma}_t(\mathrm{K}) \quad t \in \mathbb{R}$$

implies that K commutes with $\Delta_0 \otimes \Delta_\rho$.

3. It remains to show that ad K leaves invariant the weight $\omega_0 \otimes \rho_1$. By assumption we conclude that the weight $\omega_\mathrm{U} \otimes \rho_1(\,.\,)$ on $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ has the modular automorphism group

$$\mathrm{ad}(\Delta^{it} \otimes \Delta_\rho^{it}) = \mathrm{ad}((\mathrm{U}^* \otimes \mathrm{I}_\rho)(\Delta_0^{it} \otimes \Delta_\rho^{it})(\mathrm{U} \otimes \mathrm{I}_\rho)).$$

On the other hand, we infer that the weight $\omega_0 \otimes \rho_1(\mathrm{V}_\rho^*\mathrm{W}_\rho^* \cdot \mathrm{W}_\rho\mathrm{V}_\rho) = \omega_0 \otimes \rho_1(\mathrm{V}_\rho^* \cdot \mathrm{V}_\rho)$ on $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ has the modular automorphism group $\mathrm{ad}((\mathrm{W}_\rho\mathrm{V}_\rho)(\Delta_0^{it} \otimes \Delta_\rho^{it})(\mathrm{V}_\rho^*\mathrm{W}_\rho^*))$ (cf. Corollary 2.1.5). On The Other Hand,

$$\begin{aligned}\mathrm{ad}(\mathrm{W}_\rho\mathrm{V}_\rho(\Delta_0^{it} \otimes \Delta_\rho^{it})\mathrm{V}_\rho^*\mathrm{W}_\rho^*) &= \mathrm{ad}(\mathrm{W}_\rho\mathrm{V}_\rho\mathrm{K}^*(\Delta_0^{it} \otimes \Delta_\rho^{it})\mathrm{K}\mathrm{V}_\rho^*\mathrm{W}_\rho^*) \\ &= \mathrm{ad}((\mathrm{U}^* \otimes \mathrm{I}_\rho)(\Delta_0^{it} \otimes \Delta_\rho^{it})(\mathrm{U} \otimes \mathrm{I}_\rho))\end{aligned}$$

because K commutes with $\Delta_0 \otimes \Delta_\rho$. This implies that the two weights $\omega_\mathrm{U} \otimes \rho_1(\cdot)$ and $\omega_0 \otimes \rho_1(\mathrm{V}_\rho^* \cdot \mathrm{V}_\rho)$ have the same modular automorphism group. Using $\mathrm{U} \otimes \mathrm{I}_\rho = \mathrm{K}\mathrm{V}_\rho^*\mathrm{W}_\rho^*$ this gives that the weights $\omega_0 \otimes \rho_1(\cdot)$ and $\omega_0 \otimes \rho_1(\mathrm{K} \cdot \mathrm{K}^*)$ have the same modular automorphism group, i.e. they are equal up to a number $c > 0$ because $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ is a factor (cf. [Str81, 4.11]). A similar reasoning implies that ad K leaves invariant the weight $\omega_0 \otimes \rho_1$ on $\mathcal{M}_0' \otimes \mathcal{F}_\rho$.

4. We now suppose that there is a unitary operator U on $\mathcal{H}$ with the properties (ii) and (iii). We have to show that then $\mathrm{U}^* u_0$ is a cyclic and separating vector for $\mathcal{M}_0$ and $\mathrm{U}^*\Delta_0\mathrm{U}$ is the modular operator for $(\mathcal{M}_0, \mathrm{U}^* u_0)$.

   We first prove that $\mathrm{U}^* u_0$ is a cyclic and separating vector for $\mathcal{M}_0$. The assumptions imply that

   $$c\omega_{\mathrm{U}} \otimes \rho_1(\cdot) = \omega_0 \otimes \rho_1(\mathrm{Y}_1 \cdot \mathrm{Y}_1^*)$$

   on $\mathcal{M}_0 \otimes \mathcal{F}_\rho$. The right hand side is an n. s. f. weight because $\omega_0 \otimes \rho_1$ is it and Y_1 is a unitary element from $\mathcal{M}_0 \otimes \mathcal{F}_\rho$. Thus the left hand side is also such a weight. This weight is a tensor product of weights, $\omega_{\mathrm{U}} \otimes \rho_1(\cdot)$, on the tensor product $\mathcal{M}_0 \otimes \mathcal{F}_\rho$. Since the second factor ρ_1 and the tensor product $\omega_{\mathrm{U}} \otimes \rho_1(\cdot)$ are n. s. f. weights the first factor is also such a weight. This means that the vector state $\langle \cdot \mathrm{U}^* u_0 | \mathrm{U}^* u_0 \rangle$ is a faithful state on $\mathcal{M}_0$. This implies that $\mathrm{U}^* u_0$ is a separating vector for $\mathcal{M}_0$. A similar reasoning gives that the vector state $\langle \cdot \mathrm{U}^* u_0 | \mathrm{U}^* u_0 \rangle$ is a faithful state on $\mathcal{M}_0'$. This implies that $\mathrm{U}^* u_0$ is a separating vector for $\mathcal{M}_0'$. Thus $\mathrm{U}^* u_0$ is a cyclic and separating vector for $\mathcal{M}_0$.

   Second, we prove that $\mathrm{U}^*\Delta_0\mathrm{U}$ is the modular operator for $(\mathcal{M}_0, \mathrm{U}^* u_0)$. The modular automorphism group of the weight $\omega_0 \otimes \rho_1$ on $\mathcal{M}_0 \otimes \mathcal{F}_\rho$ is $\mathrm{ad}(\Delta_0^{it} \otimes \Delta_\rho^{it})$. Therefore, the modular automorphism group of the weight $\omega_{\mathrm{U}} \otimes \rho_1(\cdot) = \omega_0 \otimes \rho_1(\mathrm{Y}_1 \cdot \mathrm{Y}_1^*)$ is given by

   $$\begin{aligned}\mathrm{ad}(\mathrm{Y}_1^*(\Delta_0^{it} \otimes \Delta_\rho^{it})\mathrm{Y}_1) &= \mathrm{ad}(\mathrm{Y}_2(\mathrm{U}^* \otimes \mathrm{I}_\rho)\mathrm{K}(\Delta_0^{it} \otimes \Delta_\rho^{it})\mathrm{K}^*(\mathrm{U} \otimes \mathrm{I}_\rho)\mathrm{Y}_2^*) \\ &= \mathrm{ad}((\mathrm{U}^* \otimes \mathrm{I}_\rho)(\Delta_0^{it} \otimes \Delta_\rho^{it})(\mathrm{U} \otimes \mathrm{I}_\rho)) \\ &= \mathrm{ad}(\mathrm{U}^*\Delta_0^{it}\mathrm{U} \otimes \Delta_\rho^{it}).\end{aligned}$$

   Since $\mathrm{ad}\ \Delta_\rho^{it}$ is the modular automorphism group corresponding to the weight ρ_1 for $\mathcal{F}_\rho$ we find that $\mathrm{ad}\,\Delta^{it} = \mathrm{ad}\,\mathrm{U}^*\Delta_0^{it}\mathrm{U}$ is the modular automorphism group for the state $\langle \cdot \mathrm{U}^* u_0 | \mathrm{U}^* u_0 \rangle$ on $\mathcal{M}_0$. Furthermore, Δ^{it} is even the modular group for $(\mathcal{M}_0, \mathrm{U}^* u_0)$ because $\Delta^{it}\mathrm{U}^* u_0 = \mathrm{U}^*\Delta_0^{it} u_0 = \mathrm{U}^* u_0$. □

Remark B.0.2. 1. The similarity of the decomposition $\mathrm{U} \otimes \mathrm{I}_\rho = \mathrm{K}\mathrm{Y}_1\mathrm{Y}_2$ in (5.2.3) to the decomposition $\mathrm{U} = \mathrm{K}\mathrm{V}$ in (5.2.2) becomes closer if we restrict ourselves to type I factors $\mathcal{M}_0$. In this case the automorphism $\mathrm{ad}\,\mathrm{V}$ is inner and thus $\mathrm{V} = \mathrm{Y}_1\mathrm{Y}_2$ with $\mathrm{Y}_1 \in \mathcal{M}_0, \mathrm{Y}_2 \in \mathcal{M}_0'$.

2. Note that the decomposition $\mathrm{U} \otimes \mathrm{I}_\rho = \mathrm{K}\mathrm{Y}_1\mathrm{Y}_2$ for a given unitary U is not unique.

3. The results of the theorem also describe partly the solution of the inverse problem for the modular objects. A similar theorem for this inverse problem can be proved. However, it includes further properties of U. We omit the details.

Bibliography

[Ara72] H. Araki. Remarks on spectra of modular operators of von Neumann algebras. *Commun. Math. Phys*, 28:267–277, 1972.

[BDFS00] D. Buchholz, O. Dreyer, M. Florig, and S. J. Summers. Geometric modular action and the spacetime symmetry groups. *Rev. Math. Phys.*, 12:475–560, 2000.

[BMS00] D. Buchholz, J. Mund, and S. J. Summers. Transplantation of local nets and geometric modular action on Robertson-Walker spacetimes. arXiv:hep-th/0011237, 2000.

[Bol00a] St. Boller. Characterization of cyclic and separating vectors and application to an inverse problem in modular theory, I. Finite factors. Preprint NTZ 4/2000 and math.OA/0003087, Leipzig, 2000.

[Bol00b] St. Boller. Characterization of cyclic and separating vectors and application to an inverse problem in modular theory, II. Semifinite factors. Preprint NTZ 5/2000 and math.OA/0004030, Leipzig, 2000.

[Bor93] H. J. Borchers. A noncommuting realization of Minkowski space. In H. Araki et al., editors, *Quantum and Non-Commutative Analysis*, pages 11–30. Kluwer Dordrecht, 1993.

[Bor00] H. J. Borchers. On revolutionizing quantum field theory with Tomita's modular theory. *J. Math. Phys.*, 41(6):3604–3673, 2000.

[BR81] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics*, volume II. Springer, Berlin, 1981.

[BS87] M. S. Birman and M. Z. Solomjak. *Spectral Theory of Self-Adjoint Operators in Hilbert Space.* D. Reidel Publishing Company, Dordrecht, 1987.

[BS93] D. Buchholz and S. J. Summers. An algebraic characterization of vacuum states in Minkowski space. *Commun. Math. Phys.*, 155:449–458, 1993.

[Buc78] D. Buchholz. On the structure of local quantum fields with non-trivial interactions. In *Proceedings of the International Conference on Operator Algebras, Ideals and their Applications in Theoretical*

Pysics, Leipzig, 1977, Teubner-Texte zur Mathematik, pages 146–153, 1978.

[BW75] J. Bisognano and E. H. Wichmann. On the duality condition for a Hermitean scalar field. *J. Math. Phys.*, 16:985–1007, 1975.

[BW76] J. Bisognano and E. H. Wichmann. On the duality condition for quantum fields. *J. Math. Phys.*, 17:303–321, 1976.

[BW92] H. Baumgärtel and M. Wollenberg. *Causal Nets of Operator Algebras.* Akademie Verlag, Berlin, 1992.

[BW01] St. Boller and M. Wollenberg. An inverse problem in Tomita-Takesaki modular theory. Preprint NTZ 17/2001, Leipzig, 2001.

[BY99] H. J. Borchers and J. Yngvason. Modular groups of quantum fields in thermal states. *J. Math. Phys.*, 40:601–624, 1999.

[CFW81] A. Connes, J. Feldman, and B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergodic Theory Dynamical Systems*, 1:431–450, 1981.

[CJ82] A. Connes and V. F. R. Jones. A II_1 factor with two nonconjugate Cartan subalgebras. *Bull. Amer. Math. Soc. (N.S.)*, 6(2):211–212, 1982.

[Con73] A. Connes. Une classification des facteurs de type III. *Ann. Ec. Norm. Sup.*, 6:133–252, 1973.

[Con74] A. Connes. Almost periodic states and factors of type III_1. *J. Funct. Anal.*, 16:415–445, 1974.

[Con75] A. Connes. Outer conjugacy classes of automorphisms of factors. *Ann. Scient. Éc. Norm. Sup.*, 8:383–420, 1975.

[Con76] A. Connes. Classification of injective factors. *Ann. Math.*, 104:73–115, 1976.

[Con77] A. Connes. Periodic automorphisms of the hyperfinite factor of type II_1. *Acta Sci. Math.*, 39:39–66, 1977.

[Con85] A. Connes. Factors of type III_1, property L_λ' and closure of inner automorphisms. *J. Operator Theory*, 14:189–211, 1985.

[Con94] A. Connes. *Noncommutative Geometry.* Academic Press, San Diego, 1994.

[CS75] A. Connes and E. Størmer. Entropy for automorphisms of II_1 von Neumann algebras. *Acta Math.*, 134:289–306, 1975.

[Dix54] J. Dixmier. Sous anneaux abéliens maximaux dans les facteurs de type fini. *Ann. of Math. (2)*, 59:279–286, 1954.

[Dye52] H. A. Dye. The Radon-Nikodym theorem for finite rings of operators. *Trans. Amer. Math. Soc.*, 72:243–280, 1952.

[EK98] D. E. Evans and Y. Kawahigashi. *Quantum Symmetries on Operator Algebras.* Oxford Science Publications, 1998.

[Haa79a] U. Haagerup. L^p-spaces associated with an arbitrary von Neumann algebra. In *Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977)*, pages 175–184. CNRS, Paris, 1979.

[Haa79b] U. Haagerup. Operator valued weights in von Neumann algebras, I. *J. Funct. Anal.*, 32:175–206, 1979.

[Haa79c] U. Haagerup. Operator valued weights in von Neumann algebras, II. *J. Funct. Anal.*, 33:339–361, 1979.

[Haa87] U. Haagerup. Connes' bicentralizer problem and the uniqueness of the injective factor of type III_1. *Acta Math.*, 158:95–148, 1987.

[Haa92] R. Haag. *Local Quantum Physics.* Springer Verlag, Berlin, 1992.

[HHW67] R. Haag, N. M. Hugenholtz, and M. Winnink. On the equilibrium states in quantum statistical mechanics. *Commun. Math. Phys.*, 5:215, 1967.

[HL82] P. D. Hislop and R. Longo. Modular structure of the local algebra associated with a free massless scalar field theory. *Commun. Math. Phys.*, 84:71–85, 1982.

[IK94] A. Inoue and W. Karwowski. Cyclic generalized vectors for algebras of unbounded operators. *Publ. RIMS*, 30:577–601, 1994.

[IKO99] A. Inoue, W. Karwowski, and H. Ogi. Standard weights on algebras of unbounded operators. *J. Math. Soc. Japan*, 51(4):911–935, 1999.

[Ino95a] A. Inoue. O^*-algebras in standard system. *Math. Nachr.*, 172:171–190, 1995.

[Ino95b] A. Inoue. Standard generalized vectors for algebras of unbounded operators. *J. Math. Soc. Japan*, 47(2):329–347, 1995.

[Ino97] A. Inoue. Standard systems for semifinite O^*-algebgras. *Proc. Amer. Math. Soc.*, 125(11):3303–3312, 1997.

[Ino98] A. Inoue. *Tomita-Takesaki theory in algebras of unbounded operators.* Springer-Verlag, Berlin, 1998.

[Iso95] T. Isola. Modular structure of the crossed product by a compact group dual. *J. Operator Theory*, 33:3–31, 1995.

[Jon83] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72:1–25, 1983.

[Jon91] V. F. R. Jones. *Subfactors and Knots*, volume 80 of *Regional Conference Series in Mathematics*. American Mathematical Society, cbms edition, 1991.

[JT84] V. F. R. Jones and M. Takesaki. Actions of compact abelian groups on semifinite injective factors. *Acta Math.*, 153:213–258, 1984.

[Kaw89] Y. Kawahigashi. Centrally ergodic one-parameter automorphism groups on semifinite injective von Neumann algebras. *Math. Scand.*, 64:285–299, 1989.

[Kaw91a] Y. Kawahigashi. One-parameter automorphism groups of the hyperfinite type II_1 factor. *J. Operator Theory*, 25:37–59, 1991.

[Kaw91b] Y. Kawahigashi. One-parameter automorphism groups of the injective factor of type II_1 with Connes spectrum zero. *Can. J. Math.*, 43(1):108–118, 1991.

[Kos86] H. Kosaki. Extensions of Jones' theory on index to arbitrary factors. *J. Funct. Anal.*, 66(1986):123–140, 1986.

[KR83] R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras*, volume I. Academic Press, New York, 1983.

[KR86] R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras*, volume II. Academic Press, New York, 1986.

[KST98] Y. Katayama, C. E. Sutherland, and M. Takesaki. The characteristic square of a factor and the coycle conjugacy of discrete group actions on factors. *Invent. Math.*, 132:331.380, 1998.

[KT92] Y. Kawahigashi and M. Takesaki. Compact abelian groups on injective factors. *J. Funct. Anal.*, 105:112–128, 1992.

[LMW00] M. Leitz-Martini and M. Wollenberg. Notes on modular conjugations of von neumann factors. *Z. Anal. Anw.*, 19:13–22, 2000.

[Ocn85] A. Ocneanu. *Actions of discrete amenable groups on factors*, volume 1138 of *Lecture Notes in Math.* Springer, Berlin, 1985.

[Pop83] S. Popa. Singular maximal abelian $*$-subalgebras in continuous von Neumann algebras. *J. Funct. Anal.*, 50:151–166, 1983.

[Pop85] S. Popa. Notes on Cartan subalgebras in type II_1 factors. *Math. Scand.*, 57:171–188, 1985.

[PT73] G. K. Pedersen and M. Takesaki. The Radon-Nikodym theorem for von Neumann algebras. *Acta Math.*, 130:53–88, 1973.

[Puk56] L. Pukanszky. Some examples of factors. *Publ. Math. Debrecen*, 4:135–156, 1956.

[Puk60] L. Pukánszky. On maximal abelian subrings of factors of type II_1. *Canad. J. Math.*, 12:289–296, 1960.

[RS61] H. Reeh and S. Schlieder. Eine Bemerkung zur Unitäräquivalenz von Lorentzinvarianten Feldern. *Nuovo Cimento*, 22:1051, 1961.

[Rud87] W. Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[Rud90] W. Rudin. *Fourier analysis on groups.* John Wiley & Sons Inc., New York, 1990. Reprint of the 1962 original, A Wiley-Interscience Publication.

[RVDV77] R. Rousseau, A. Van Daele, and L. Vanhesswijck. A necessary and sufficient condition for a von neumann algebra to be in standard form. *J. London Math. Soc.*, 15:147–154, 1977.

[Sch97] B. Schroer. Notes on the wigner representation theory of the poincare group, localization and statistics. *Nucl. Phys. B*, 499:519–546, 1997.

[Ska77] C. F. Skau. Finite subalgebras of a von Neumann algebra. *J. Funct. Anal.*, 25:211–235, 1977.

[ST89] C. E. Sutherland and M. Takesaki. Actions of discrete amenable groups on injective factors of type III_λ, $\lambda \neq 1$. *Pacific J. Math.*, 137:405–444, 1989.

[Str81] S. Strătilă. *Modular Theory in Operator Algebras.* Editura Academiei Bucuresti and Abacus Press Tunbridge Wells, 1981.

[Sun87] V. S. Sunder. *An Invitation to von Neumann Algebras.* Springer-Verlag, New York, 1987.

[SZ79] S. Strătilă and L. Zsidó. *Lectures on von Neumann Algebras.* Editura Academiei Bucuresti and Abacus Press Tunbridge Wells, 1979.

[Tak70] M. Takesaki. *Tomita's theory of modular Hilbert algebras and its applications*, volume 128 of *Lecture Notes in Mathematics.* Springer Verlag, Berlin, 1970.

[Tak83] M. Takesaki. *Structure of Factors and Automorphism Groups.* Number 51 in Regional Conference Series in Mathematics. AMS, 1983.

[Tau65] R. J. Tauer. Maximal abelian subalgebras in finite factors of type II_1. *Trans. Amer. Math. Soc.*, 114:281–308, 1965.

[Tom67] M. Tomita. Quasi-standard von Neumann algebras. Preprint, 1967.

[Wei76] J. Weidmann. *Lineare Operatoren im Hilbertraum.* Teubner, Stuttgart, 1976.

[Wei97] A. Weinstein. The modular automorphism group of a Poisson manifold. *J. Geom. Phys.*, 23(3-4):379–394, 1997.

[Wer95] Dirk Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, 1995.

[Wol92] M. Wollenberg. Notes on pertubations of causal nets of operator algebras. SFB Preprint 288 Nr.36, Berlin, 1992.

[Wol97] M. Wollenberg. An inverse problem in modular theory, I. General facts and a first answer. Preprint NTZ 32/1997, Leipzig, 1997.

[Wol98] M. Wollenberg. An inverse problem in modular theory, II. Type I factors. Preprint NTZ 5/1998, Leipzig, 1998.

[Yam92] S. Yamagami. Algebraic aspects in modular theory. *Publ. RIMS*, 28:1075–1106, 1992.

[Yam94] S. Yamagami. Modular theory for bimodules. *J. Funct. Anal.*, 125(2):327–357, 1994.

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Selbständigkeitserklärung

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Leipzig, den 06. September 2001

Stefan Boller